

AN H^2 RIEMANNIAN METRIC ON THE SPACE OF PLANAR CURVES MODULO SIMILITUDES

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ABSTRACT. Analyzing shape manifolds as Riemannian manifolds has been shown to be an effective technique for understanding their geometry. Riemannian metrics of the type H^0 and H^1 on the space of planar curves have already been investigated in detail. Here, an H^2 metric is formulated which in particular depends on the local bending of the curve rather than changes in its local orientation. Since in many applications, the basic shape of an object is understood to be independent of its scale, orientation or placement, we consider the space of planar curves modulo similitudes. Equations of the geodesic for parametrized curves as well as for unparametrized curves have been derived. Bounds for the sectional curvature are also derived.

1. INTRODUCTION

One of the approaches to shape analysis is to study the Riemannian geometry of the shape manifold. Examples of this approach may be seen in [2-6,8-15]. In particular, there has been a great deal of progress in understanding the geometry of planar shapes exemplified by silhouettes and MRI. Mathematically, this space is the space of smooth planar curves. Michor and Mumford have analyzed this space in considerable detail. In particular, they derive the geodesic equation for a general Sobolev metric. A more detailed analysis including the sectional curvature of specific metrics is carried out in [8,15]. Conformal versions of the H^0 -metric of Michor and Mumford are studied in [8] while an H^1 -metric is studied in [15]. An H^2 -metric on space of curves parametrized by the arc length has been formulated in [2].

It is useful to separate out the pose and scale of an object from its inherent geometry. This leads us to study the space of planar curves modulo translation, rotation and scale. An example of such a formulation is an H^1 -metric analyzed in [15] by Younes et al. Although rotations have been factored out, the metric still depends on local rotations of the curve. To obtain a metric which depends purely on the bending and stretching of the curve, what is needed is an H^2 -metric. In this paper, we construct such a metric and derive the geodesic equation. We also give the equation of gradient descent for deforming a given curve into a geodesic. Using the techniques developed in [3,4,15], we compute the sectional curvature and derive an absolute bound.

The results described here may be generalized to Sobolev metrics of higher order by iterating the steps for the H^2 metric. It can be shown that the sectional curvature of the space of parametrized curves tends to zero as the order of the metric tends to infinity. The limiting behavior in the case of unparametrized curves is unclear.

2. SOBOLEV METRICS

Let $\text{Imm}(S^1, \mathbb{R}^2)$ denote the space of all C^∞ immersions $c : S^1 \rightarrow \mathbb{R}^2$. Let the unit circle S^1 be parametrized by θ . A tangent vector h at a point c in $\text{Imm}(S^1, \mathbb{R}^2)$ is just a vector field in \mathbb{R}^2 along the image curve of c in \mathbb{R}^2 . Let v and n denote the vector fields of unit tangent and unit normal vectors along c and let s denote the arc-length. The infinitesimal arc-length $ds = |c_\theta| d\theta$. (Subscripts θ, s, t denote the derivative.) If a, b are vectors in \mathbb{R}^2 and $a \cdot b$ will denote their dot product. We will view a vector field h along c also as a complex function when c is viewed as a curve in the complex plane \mathbb{C} . As complex functions, $v = c_s$ and $n = ic_s$. Mixing the two notations, we get $h_s/c_s = h_s \cdot v + ih_s \cdot n$.

Let Σ denote the group of similitudes. Its action on c as curve in \mathbb{C} is given by $\alpha c + \beta$ where α, β are (complex) constants. The vertical vectors at c of the quotient map $\rho : \text{Imm}(S^1, \mathbb{R}^2) \rightarrow \text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ have the form $\alpha c + \beta$. If h is a vertical vector, then h_s/c_s is constant, that is, $h_s \cdot v$ and $h_s \cdot n$ are constant. We define Σ -horizontal vectors at c as

$$\{h | \overline{h} = \overline{h_s \cdot v} = \overline{h_s \cdot n} = 0\}$$

where the bar over a variable denotes its average value: $\overline{f} = \frac{1}{\ell} \int_c f ds$, $\ell = \int_c ds$ is the length of c . The horizontal and the vertical vectors define a decomposition of the tangent bundle of $\text{Imm}(S^1, \mathbb{R}^2)$ into two subbundles such that the tangent map $T_c \text{Imm}(S^1, \mathbb{R}^2) \rightarrow T_{\rho(c)} \text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ is an isomorphism when restricted to the horizontal vectors. We identify the tangent vectors at $\rho(c)$ with the horizontal tangent vectors at c . The projection from the tangent space at c onto the space of horizontal tangent vectors is given by the formula

$$h \mapsto h^\Sigma = (h - \overline{h}) - \overline{h_s/c_s}(c - \overline{c})$$

Given a path $c(t)$ in $\text{Imm}(S^1, \mathbb{R}^2)$, there exists a horizontal path $c^\Sigma(t) = \alpha(t)c(t) + \beta(t)$ where $\alpha(t)$ and $\beta(t)$ are given by equations

$$\begin{aligned} \alpha_t + \overline{\alpha c_{ts}/c_s} &= 0 \\ \beta_t + \overline{(\alpha c)_t} &= 0 \end{aligned}$$

such that $c(t)$ and $c^\Sigma(t)$ project to the same path in $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$.

We now define a bilinear form on the tangent space at c :

$$\langle m, h \rangle = \ell^{2n-3} \int_c (D_s^{n-1}(m_s^\Sigma \cdot v), D_s^{n-1}(m_s^\Sigma \cdot n)) \cdot (D_s^{n-1}(h_s^\Sigma \cdot v), D_s^{n-1}(h_s^\Sigma \cdot n)) ds, \quad n \geq 1$$

where $D_s = d/ds$ and ℓ is the length of c . Note that $h_s^\Sigma \cdot v = h_s \cdot v - \overline{h_s \cdot v}$ and $h_s^\Sigma \cdot n = h_s \cdot n - \overline{h_s \cdot n}$. Moreover $\langle h, h \rangle = 0$ if and only if h is a vertical vector. $\langle m, h \rangle$ is non-degenerate on the horizontal subbundle and defines a Riemannian metric on it and hence on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. The case $n = 1$ has been treated in [13] using a representation of $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ by grassmannians. Here we consider the case $n = 2$. Since $D_s^{n-1}(h_s^\Sigma \cdot v, h_s^\Sigma \cdot n) = D_s^{n-1}(h_s \cdot v, h_s \cdot n)$ if $n \geq 2$,

$$\langle m, h \rangle = \ell^{2n-3} \int_c (D_s^{n-1}(m_s \cdot v, m_s \cdot n)) \cdot (D_s^{n-1}(h_s \cdot v, h_s \cdot n)) ds, \quad n \geq 2$$

Given a path $c(t) : [0, 1] \rightarrow \text{Imm}(S^1, \mathbb{R}^2)$, we define its length by setting

$$L(c(t)) = \int_0^1 \sqrt{\langle c_t, c_t \rangle} dt$$

The length of a path is invariant under the action of Σ and equals zero if and only if the path is vertical.

3. GEODESIC EQUATION FOR THE H^2 -METRIC ON $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$

Let $c(t) : [0, 1] \rightarrow \text{Imm}(S^1, \mathbb{R}^2)$ be a lift of a path in $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. We may assume that for each t , the curve $c(t)$ has unit length. Deformation of a curve by a horizontal tangent vector preserves its length. We get the equation of a geodesic by calculating the first variation of the energy

$$E(c) = \frac{1}{2} \int_0^1 \langle c_t, c_t \rangle dt$$

Let $m(t)$ be a field of horizontal tangent vectors along $c(t)$, vanishing at its endpoints. We have the following formulas for derivatives in the direction of m :

$$\begin{aligned} D_m |c_\theta| &= (m_s \cdot v) |c_\theta|, & D_m D_s &= -(m_s \cdot v) D_s \\ D_m v &= (m_s \cdot n) n, & D_m n &= -(m_s \cdot n) v \\ D_m (c_{ts} \cdot v) &= (m_s \cdot v)_t, & D_m (c_{ts} \cdot n) &= (m_s \cdot n)_t \end{aligned}$$

$$\begin{aligned} D_m E(c) &= \frac{1}{2} \int_0^1 D_m \int_{S^1} ((c_{ts} \cdot v)_\theta^2 + (c_{ts} \cdot n)_\theta^2) \frac{d\theta}{|c_\theta|} dt \\ &= \int_0^1 \int_{S^1} (m_s \cdot v, m_s \cdot n)_{\theta t} \cdot (c_{ts} \cdot v, c_{ts} \cdot n)_\theta \frac{d\theta}{|c_\theta|} dt \\ &\quad - \frac{1}{2} \int_0^1 \int_{S^1} (m_s \cdot v) ((c_{ts} \cdot v)_\theta^2 + (c_{ts} \cdot n)_\theta^2) \frac{d\theta}{|c_\theta|} dt \\ &= - \int_0^1 \int_c (m_s \cdot v, m_s \cdot n)_s \cdot (c_{ts} \cdot v, c_{ts} \cdot n)_{st} ds dt \\ &\quad + \frac{1}{2} \int_0^1 \int_c (m_s \cdot v)_s D_s^{-1} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 ds dt \\ &= - \int_0^1 \langle m, \gamma(c) \rangle dt \end{aligned}$$

Here, the superscript 0 on a quantity means the quantity minus its average value: $f^0 = f - \bar{f}$. The operator D_s^{-1} is uniquely defined on zero-mean functions by requiring that $D_s^{-1} f = 0$. The symbol γ denotes the geodesic curvature of the path $c(t)$. An explicit expression for γ may be derived as follows. Note that

$$\begin{aligned} f_{st} &= (D_t D_s) f + D_s (D_t f) = -(c_{ts} \cdot v) f_s + f_{ts} \\ (\gamma_s \cdot v)_s &= (c_{ts} \cdot v)_{st} - \frac{1}{2} D_s^{-1} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 \\ (\gamma_s \cdot n)_s &= (c_{ts} \cdot n)_{st} \\ (c_{ts} \cdot v)_{st} &= (c_{ts} \cdot v)_{ts} - (c_{ts} \cdot v)(c_{ts} \cdot v)_s \\ &= (c_{tts} \cdot v - (c_{ts} \cdot v)^2 + (c_{ts} \cdot n)^2)_s - (c_{ts} \cdot v)(c_{ts} \cdot v)_s \\ &= \left(c_{tts} \cdot v - \frac{3}{2} (c_{ts} \cdot v)^2 + (c_{ts} \cdot n)^2 \right)_s \end{aligned}$$

$$\begin{aligned} (c_{ts} \cdot n)_{st} &= (c_{ts} \cdot n)_{ts} - (c_{ts} \cdot v)(c_{ts} \cdot n)_s \\ &= (c_{tts} \cdot n - 2(c_{ts} \cdot v)(c_{ts} \cdot n))_s - (c_{ts} \cdot v)(c_{ts} \cdot n)_s \end{aligned}$$

Let Λ be the complex function

$$\begin{aligned} \Lambda &= \left[\left(\frac{3}{2}(c_{ts} \cdot v)^2 - (c_{ts} \cdot n)^2 \right) + \frac{1}{2} D_s^{-2} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 \right] \\ &\quad + i \left[(2(c_{ts} \cdot v)(c_{ts} \cdot n)) + D_s^{-1} ((c_{ts} \cdot v)(c_{ts} \cdot n)_s)^0 \right] \end{aligned}$$

Then,

$$(\gamma_s/c_s)_s = (\gamma_s \cdot v)_s + i(\gamma_s \cdot n)_s = \left(\frac{c_{tts}}{c_s} - \Lambda \right)_s$$

$$\begin{aligned} D_m E(c) &= - \int_0^1 \int_c \left(\frac{m_s}{c_s} \right)_s \cdot \left(\frac{\gamma_s}{c_s} \right)_s ds \\ &= -\Re \int_0^1 \int_c \left(\frac{m_s}{c_s} \right)_s^* \left(\frac{\gamma_s}{c_s} \right)_s ds \end{aligned}$$

where the superscript $*$ denotes the complex conjugate. Since $\int_c m_s ds = 0$, we have

$$0 = \int_c (m_s/c_s)^* c_s^* ds = \int_c (m_s/c_s)_s^* D_s^{-1}(c_s^*) ds$$

and

$$D_m E(c) = -\Re \int_0^1 \int_c (m_s/c_s)_s^* [(c_{tts}/c_s) - \Lambda - \alpha D_s^{-2}(c_s^*)]_s ds$$

for all $\alpha \in \mathbb{C}$. We may write

$$\gamma_s/c_s = (c_{tts}/c_s) - \Lambda - \alpha D_s^{-2}(c_s^*)$$

Since we must have $\int_c \gamma_s ds = 0$, set

$$\alpha = \frac{\int_c \Lambda c_s ds}{\int_c |D_s^{-1}(c_s)|^2 ds}$$

Finally,

$$\gamma = c_{tt} - D_s^{-1} (\Lambda + \alpha D_s^{-2}(c_s^*)) c_s^0$$

A path $c(t)$ is a geodesic if and only if $\gamma(c) = 0$. A path $c(t)$ may be deformed into a geodesic by gradient descent: $\frac{\partial c}{\partial \tau} \propto \gamma$.

An alternative form of the geodesic equation may be obtained by integration by parts.

$$\begin{aligned} \langle m, \gamma \rangle &= - \int_c ((m_s \cdot v)(\gamma_s \cdot v)_{ss} + (m_s \cdot n)(\gamma_s \cdot n)_{ss}) ds \\ &= - \int_c ((m_s \cdot v)v + (m_s \cdot n)n) \cdot ((\gamma_s \cdot v)_{ss}v + (\gamma_s \cdot n)_{ss}n) ds \\ &= - \int_c m_s \cdot ((\gamma_s \cdot v)_{ss}v + (\gamma_s \cdot n)_{ss}n) ds \\ &= \int_c m \cdot ((\gamma_s \cdot v)_{ss}v + (\gamma_s \cdot n)_{ss}n)_s ds \end{aligned}$$

for $m \in T_c \text{Imm}(S^1, \mathbb{R}^2)$. The geodesic equation may be written as

$$\left((c_{ts} \cdot v)_{sts} v + (c_{ts} \cdot n)_{sts} n - \frac{1}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 v \right)_s = 0$$

Since $\langle ((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s^\Sigma, \gamma \rangle = \langle ((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s, \gamma \rangle \geq 0$, the gradient descent equation takes form

$$\frac{\partial c}{\partial \tau} \propto \left((c_{ts} \cdot v)_{sts} v + (c_{ts} \cdot n)_{sts} n - \frac{1}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 v \right)_s^\Sigma$$

The geodesic equation may also be written in terms of the momentum vector u :

$$u = ((c_{ts} \cdot v)_{ss} v + (c_{ts} \cdot n)_{ss} n)_s$$

As a complex function, $u = ((c_{ts}/c_s)_{ss} c_s)_s$.

$$\begin{aligned} u_t &= ((c_{ts}/c_s)_{ss} c_s)_{st} \\ &= -(c_{ts} \cdot v) u + ((c_{ts}/c_s)_{ss} c_s)_{ts} \\ &= -(c_{ts} \cdot v) u + ((c_{ts}/c_s)_{sst} c_s)_s + i((c_{ts} \cdot n) (c_{ts}/c_s)_{ss} c_s)_s \\ &= -(c_{ts} \cdot v) u + (- (c_{ts} \cdot v) (c_{ts}/c_s)_{ss} c_s + (c_{ts}/c_s)_{sts} c_s)_s \\ &\quad + i((c_{ts} \cdot n) u + (c_{ts} \cdot n)_s (c_{ts}/c_s)_{ss} c_s) \end{aligned}$$

Therefore,

$$\begin{aligned} u_t &= (-2c_{ts} \cdot v + ic_{ts} \cdot n) u \\ &\quad + (-c_{ts} \cdot v + ic_{ts} \cdot n)_s (c_{ts}/c_s)_{ss} c_s + ((c_{ts}/c_s)_{sts} c_s)_s \end{aligned}$$

Substituting the geodesic equation

$$0 = ((\gamma_s/c_s)_{ss} c_s)_s = ((c_{ts}/c_s)_{sts} c_s)_s - \frac{1}{2} \left((\| (c_{ts}/c_s)_s \|^2)^0 c_s \right)_s$$

we get

$$\begin{aligned} &u_t + (2c_{ts} \cdot v - ic_{ts} \cdot n) u \\ &= (-c_{ts} \cdot v + ic_{ts} \cdot n)_s (c_{ts} \cdot v + ic_{ts} \cdot n)_{ss} c_s \\ &\quad + \frac{1}{2} \left((\| (c_{ts}/c_s)_s \|^2)^0 c_s \right)_s \\ &= i[(c_{ts} \cdot v)_{ss} (c_{ts} \cdot n)_s - (c_{ts} \cdot v)_s (c_{ts} \cdot n)_{ss} \\ &\quad + \frac{\kappa}{2} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0] c_s \end{aligned}$$

where κ is the curvature of c defined by the equation $v_s = \kappa n$.

4. GEODESIC EQUATION FOR THE H^2 -METRIC ON $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$

The group of diffeomorphisms $\text{Diff}(S^1)$ acts on $\text{Imm}(S^1, \mathbb{R}^2)$ by composition from the right. The action commutes with the action of Σ and hence $\text{Diff}(S^1)$ acts on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. The quotient $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ is the space of unparametrized curves modulo similitudes which inherits a Riemannian metric making the quotient map a submersion. The tangent bundle of $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ splits into vertical and horizontal subbundles which are orthogonal with respect to the metric on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. The geodesics on $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ are just the horizontal geodesics on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. If a Σ -horizontal path in $\text{Imm}(S^1, \mathbb{R}^2)$ is also horizontal with respect to $\text{Diff}(S^1)$, then its geodesic curvature γ is $\text{Diff}(S^1)$ -horizontal as well. The action of an infinitesimal diffeomorphism corresponds to a

vector field bv along c where $b \in \mathbb{R}$. A tangent vector $h \in T_{\rho(c)} \text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ is horizontal

$$\begin{aligned} \iff \langle bv, h \rangle &= \int_c bv \cdot ((h_s \cdot v)_{ss} v + (h_s \cdot n)_{ss} n)_s ds = 0 \\ \iff v \cdot ((h_s \cdot v)_{ss} v + (h_s \cdot n)_{ss} n)_s &= (h_s \cdot v)_{sss} - \kappa (h_s \cdot n)_{ss} = 0 \end{aligned}$$

Therefore, γ satisfies the equation $((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s \cdot v = 0$ so that

$$\begin{aligned} &((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s \\ &= \left((c_{ts} \cdot v)_{sts} \kappa + (c_{ts} \cdot n)_{stss} - \frac{\kappa}{2} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 \right) n \end{aligned}$$

The geodesic equation takes the form

$$(c_{ts} \cdot v)_{sts} \kappa + (c_{ts} \cdot n)_{stss} = \frac{\kappa}{2} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0$$

Now if a path $c(t)$ is horizontal with respect to $\text{Diff}(S^1)$, then $u \cdot v = 0$ so that $u = an$, $a(t) \in \mathbb{R}$. Therefore, in terms the momentum u , the geodesic equation takes the form

$$\begin{aligned} a_t + 2(c_{ts} \cdot v)a &= (c_{ts} \cdot v)_{ss} (c_{ts} \cdot n)_s - (c_{ts} \cdot v)_s (c_{ts} \cdot n)_{ss} \\ &\quad + \frac{\kappa}{2} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 \end{aligned}$$

We may construct geodesics in $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ as follows. Given a path $C(t)$ in $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$, lift it to a path in $\text{Imm}(S^1, \mathbb{R}^2)$ which is horizontal with respect to Σ as well as $\text{Diff}(S^1)$ and then deform it by the gradient descent equation

$$\frac{\partial c}{\partial \tau} \propto \left((c_{ts} \cdot v)_{sts} \kappa + (c_{ts} \cdot n)_{stss} - \frac{\kappa}{2} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 \right)^\Sigma$$

A horizontal path $c(t)$ may be constructed as follows. Lift $C(t)$ to a path $c(t)$ on $\text{Imm}(S^1, \mathbb{R}^2)$. We now construct a path $\varphi(t, \theta) \in \text{Diff}(S^1)$ such that $c \circ \varphi$ is horizontal. Write $\varphi_t(t, \theta) = \xi(t, \theta) \circ \varphi(t, \theta)$ and let $\eta = |c_\theta| \xi$. Note that $(c \circ \varphi)_t = (c_t + \eta v) \circ \varphi$ and $D_{c \circ \varphi}(f \circ \varphi) = (D_c f) \circ \varphi$ (see [3]). The path $c \circ \varphi$ is horizontal with respect to $\text{Diff}(S^1)$

$$\begin{aligned} \iff &((c_t + \eta v)_s \cdot v)_{sss} - \kappa ((c_t + \eta v)_s \cdot n)_{ss} = 0 \\ \iff &L\eta = -(c_{ts} \cdot v)_{sss} + \kappa (c_{ts} \cdot n)_{ss} \end{aligned}$$

where $L\eta = \eta_{ssss} - \kappa(\kappa\eta)_{ss}$. The operator L is self-adjoint and it is positive definite provided that c is not a circle. Assuming that the path does not pass through a circle, η may be calculated by minimizing

$$\int_c \frac{1}{2} (\eta_{ss}^2 + (\kappa\eta)_s^2) + ((c_{ts} \cdot v)_{sss} - \kappa(c_{ts} \cdot n)_{ss}) \eta ds$$

by gradient descent. Then, integrate the equation $\varphi_t(t, \theta) = \xi(t, \theta) \circ \varphi(t, \theta)$ to obtain φ and the horizontal path $(c \circ \varphi)^\Sigma$.

5. SECTIONAL CURVATURE

5.1. **Local Charts.** Let $c : S^1 \rightarrow \mathbb{C}$ be an immersion. c_θ is translation invariant.

Σ acts on c_θ by multiplication and on $\log c_\theta$ by translation. Hence, $(\log c_\theta)_\theta$ is Σ -invariant.

$$\int_{S^1} (\log c_\theta)_\theta d\theta = \log c_\theta|_{2\pi} - \log c_\theta|_0 = 2\pi Ji$$

where J is the rotation index of c . The space of $(\log c_\theta)_\theta$ provides local charts of $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ as follows.

Fix an integer J , the rotation index. Let

$$\begin{aligned} \Omega &= \left\{ z \in C^\infty(S^1, \mathbb{R}^2) \mid \int_{S^1} z_1 d\theta = 0, \int_{S^1} z_2 d\theta = 2\pi J \right\} \\ \tilde{\Omega} &= \left\{ z \in C^\infty(\mathbb{R}, \mathbb{R}^2) \mid z_1, z_2 - Jx \text{ } 2\pi\text{-periodic, } \int_0^{2\pi} e^{z_1} dx = 1 \right\} \end{aligned}$$

where z_1, z_2 are the components of z . The map $\chi : \tilde{\Omega} \rightarrow \Omega$ is defined by setting $\chi(z) = dz/dx$. A $z \in \tilde{\Omega}$ defines a curve $c_z : \mathbb{R} \rightarrow \mathbb{C}$:

$$c_z(x) = \int_0^x e^{z_1 + iz_2} dx$$

c_z is a closed curve of length 1 if $c_z(2\pi) = 0$. The arclength s of c_z is given by $s(x) = \int_0^x e^{z_1} dx$. The unit tangent vector field v and the unit normal vector field n along c_z are given by e^{iz_2} and ie^{iz_2} . The curvature κ of the curve c_z is given by $\kappa = z_2 s^{-1}$.

Let $\tilde{\Omega}_0 = \{z : c_z \text{ is closed}\}$. Let $\Omega_0 = \chi(\tilde{\Omega}_0)$. Define a map $\Omega_0 \rightarrow \text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ as follows. Let $\zeta \in \Omega_0$. $\chi^{-1}(\zeta)$ consists of pairs (z_1, z_2) such that z_1 is uniquely determined and z_2 is unique upto a constant. Let $c_{\chi^{-1}(\zeta)} = \{c_z : z \in \chi^{-1}(\zeta)\}$. The curves $\chi^{-1}(\zeta)$ lie in the same Σ -orbit and hence map to a unique point in $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$.

5.2. **Tangent Bundles.** If a function $f(x)$ is 2π -periodic, we will regard it as a function on S^1 as well as a function on the circle $\Delta = \{e^{2\pi is} : 0 \leq s \leq 1\}$. Let $z \in \tilde{\Omega}$. The tangent spaces $T_z \tilde{\Omega}$ and $T_{\chi(z)} \Omega$ are given by

$$T_z \tilde{\Omega} = \{h = (h_1, h_2) : h_1, h_2 \text{ } 2\pi\text{-periodic, } \int_{\Delta} h_1 ds = 0\}$$

$$T_{\chi(z)} \Omega = \{h_\theta : h \in T_z \tilde{\Omega}\}$$

The vertical vectors at z are of the form $(0, a)$, $a \in \mathbb{R}$. Let

$$T_z^0 \tilde{\Omega} = \{h : \int_{\Delta} h ds = 0\}$$

We have the decomposition $T_z \tilde{\Omega} \simeq T_z^0 \tilde{\Omega} \oplus \mathbb{R}$ and $T_z^0 \tilde{\Omega} \simeq T_{\chi(z)} \Omega$. Every tangent vector field on Ω lifts uniquely to a section of $T^0 \tilde{\Omega}$. We will carry out all the calculations below in terms of the sections of $T^0 \tilde{\Omega}$.

Let $m \in T_c \text{Imm}(S^1, \mathbb{R}^2)$. Then,

$$\begin{aligned} D_m c_\theta &= D_m(|c_\theta|v) \\ &= (m_s \cdot v)|c_\theta|v + (m_s \cdot n)n|c_\theta| \\ &= \{(m_s \cdot v) + i(m_s \cdot n)\}c_\theta \end{aligned}$$

and $D_m \log c_\theta = (m_s \cdot v) + i(m_s \cdot n)$. Therefore, if z_c is the image of c in $\tilde{\Omega}$, m maps onto $(m_s \cdot v, m_s \cdot n) \in T_{z_c} \tilde{\Omega}$. The subRiemannian metric on $\text{Imm}(S^1, \mathbb{R}^2)$ induces a subRiemannian metric on $\tilde{\Omega}$. If $m, h \in T_z \tilde{\Omega}$,

$$\langle m, h \rangle = \int_{\Delta} m_s \cdot h_s ds$$

The metric is non-degenerate on $T^0 \tilde{\Omega}$ and induces a Riemannian metric on Ω .

5.3. Action of the Group of Reparametrization. Let $\text{Diff}^+(\mathbb{R})$ be the group of increasing C^∞ diffeomorphisms $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi(x+2\pi) = \varphi(x) + 2\pi$ for all x . Let $\text{Diff}^0(\mathbb{R})$ be the central subgroup of translations: $\varphi(x) = x + 2\pi n$. The quotient $\text{Diff}^+(\mathbb{R}) / \text{Diff}^0(\mathbb{R})$ is the group of diffeomorphisms of S^1 , $\text{Diff}^+(S^1)$. $\text{Diff}^+(\mathbb{R})$ acts on $\tilde{\Omega}$ as follows: $z_1^\varphi = z_1 \circ \varphi + \log \varphi_x$; $z_2^\varphi = z_2 \circ \varphi$. The action commutes with the translations $z_2 \rightarrow z_2 + a$ and hence $\text{Diff}^+(\mathbb{R})$ acts on Ω . The action of $\text{Diff}^0(\mathbb{R})$ on Ω is trivial so that $\text{Diff}^+(S^1)$ acts on Ω . The metric on Ω is defined independently from parametrization. Therefore the quotient map $\Pi : \Omega \rightarrow \Omega / \text{Diff}^+(S^1)$ is a Riemannian submersion. An infinitesimal diffeomorphism $a \in \mathbb{R}$ of S^1 induces a tangent vector field bv along c_z where $b = e^{z_1} a$ and maps to a vertical vector $(L_1 b)^0 \in T_z^0 \tilde{\Omega}$ where $L_1 = (D_s, \kappa)$. A tangent vector $h \in T_z^0 \tilde{\Omega}$ is Π -horizontal $\Leftrightarrow \langle h, (L_1 b)^0 \rangle = 0 \Leftrightarrow L_1^* \cdot h_{ss} = 0$ where the adjoint $L_1^* = (-D_s, \kappa)$. Let h^V and h^H denote the horizontal and vertical components of h . h^V has the form $(L_1 b)^0$. Since $L_1^* \cdot (h - L_1 b)_{ss} = 0$,

$$L_1^* \cdot h_{ss} = L_1^* \cdot (L_1 b) = -b_{ssss} + \kappa(\kappa b)_{ss} = -Lb$$

where L is the self-adjoint operator defined earlier. We have

$$h^V = - (L_1 \cdot L^{-1} (L_1^* \cdot h_{ss}))^0$$

provided that c_z is not a circle.

5.4. Christoffel Symbols. Let h, k, m be sections of $T^0 \tilde{\Omega}$ such that $h_\theta, k_\theta, m_\theta$ are constant vector fields on Ω . We calculate the Christoffel symbol $\Gamma(h, k) \in T^0 \tilde{\Omega}$ using the two gradients, $K(m, h)$ and $H(h, k)$ of $D_m \langle h, k \rangle$:

$$D_m \langle h, k \rangle = \langle K(m, h), k \rangle = \langle m, H(h, k) \rangle$$

$$\begin{aligned} \int_{\Delta} K_s(m, h) \cdot k_s ds &= D_m \int_{\Delta} h_s \cdot k_s ds \\ &= D_m \int_{S^1} h_\theta \cdot k_\theta e^{-z_1} d\theta \\ &= - \int_{S^1} h_\theta \cdot k_\theta m_1 e^{-z_1} d\theta \\ &= - \int_{\Delta} h_s \cdot k_s m_1 ds \\ &\quad - \int_{\Delta} (m_1 h_s)^0 \cdot k_s ds \end{aligned}$$

Therefore, $K_s(m, h) = -(m_1 h_s)^0$. Similarly,

$$\begin{aligned} \int_{\Delta} H_s(h, k) \cdot m_s ds &= - \int_{\Delta} h_s \cdot k_s m_1 ds \\ &= - \int_{\Delta} (h_s \cdot k_s)^0 m_1 ds \\ &= \int_{\Delta} m_{1s} D_s^{-1} (h_s \cdot k_s)^0 ds \end{aligned}$$

Hence, $H_s(h, k) = (D_s^{-1} (h_s \cdot k_s)^0, 0)$. We have the identity

$$\Gamma(h, k) = \frac{1}{2} [K(h, k) + K(k, h) - H(h, k)]$$

Therefore,

$$\Gamma_s(h, k) = -\frac{1}{2} \left[(h_1 k_s)^0 + (k_1 h_s)^0 + (D_s^{-1} (h_s \cdot k_s)^0, 0) \right].$$

5.5. Sectional Curvature of Ω . Let m, h be sections of $T^0\tilde{\Omega}$ such that m_θ, h_θ are constant vector fields on Ω . We have the identity for the sectional curvature at the two-dimensional subspace of the tangent space at $\zeta \in \Omega$ spanned by m and h :

$$\begin{aligned} \kappa_{\zeta, \Omega}(m, h) &= D_{m, h}^2 \langle m, h \rangle - \frac{1}{2} D_{m, m}^2 \langle h, h \rangle - \frac{1}{2} D_{h, h}^2 \langle m, m \rangle \\ &\quad + \langle \Gamma(m, h), \Gamma(m, h) \rangle - \langle \Gamma(m, m), \Gamma(h, h) \rangle \end{aligned}$$

In evaluating the right-hand side, we will use the formula $D_m h = -\overline{m_1 h}$ obtained as follows: Let $\tilde{h} = \int_0^\theta h_\theta d\theta$. Then, $h = \tilde{h} - \int_{\Delta} \tilde{h} ds$. $D_m h = -\int \tilde{h} m_1 ds = -\int_{\Delta} h m_1 ds$.

Let m, h, k, p be tangent vector fields on $\tilde{\Omega}$ which are sections of $T^0\tilde{\Omega}$ and which are constant on Ω .

$$\begin{aligned} D_{m, p}^2 \langle h, k \rangle &= -D_p \int_{S^1} h_\theta \cdot k_\theta m_1 e^{-z_1} d\theta \\ &= \int_{S^1} h_\theta \cdot k_\theta m_1 p_1 e^{-z_1} d\theta + \overline{m_1 p_1} \int_{S^1} h_\theta \cdot k_\theta e^{-z_1} d\theta \\ &= \int_{\Delta} m_1 p_1 h_s \cdot k_s ds + \overline{m_1 p_1} \langle h, k \rangle \end{aligned}$$

We may assume that m, h are orthonormal at ζ . Then,

$$\begin{aligned} D_{m, h}^2 \langle m, h \rangle &= \int_{\Delta} m_1 h_1 m_s \cdot h_s ds \\ -\frac{1}{2} D_{m, m}^2 \langle h, h \rangle &= -\frac{1}{2} \int_{\Delta} m_1^2 h_s \cdot h_s ds - \frac{1}{2} \overline{m_1^2} \\ -\frac{1}{2} D_{h, h}^2 \langle m, m \rangle &= -\frac{1}{2} \int_{\Delta} h_1^2 m_s \cdot m_s ds - \frac{1}{2} \overline{h_1^2} \end{aligned}$$

The expression $\langle \Gamma(m, h), \Gamma(m, h) \rangle - \langle \Gamma(m, m), \Gamma(h, h) \rangle$ simplifies to

$$\begin{aligned} & \frac{1}{2} \int_{\Delta} |h_1 m_s - m_1 h_s|^2 ds + \\ & \frac{1}{4} \int_{\Delta} \left[\left| D_s^{-1}(m_s \cdot h_s)^0 \right|^2 - D_s^{-1}(m_s \cdot m_s)^0 D_s^{-1}(h_s \cdot h_s)^0 \right] ds \\ & - \frac{1}{4} \left(\overline{h_1 m_s + m_1 h_s}^2 + \overline{m_1^2 + h_1^2} \right) + \overline{m_1 m_s} \cdot \overline{h_1 h_s} \end{aligned}$$

Summing all the terms, we get,

$$\begin{aligned} & \kappa_{\zeta, \Omega}(m, h) \\ & = \frac{1}{4} \int_{\Delta} \left[\left| D_s^{-1}(m_s \cdot h_s)^0 \right|^2 - D_s^{-1}(m_s \cdot m_s)^0 D_s^{-1}(h_s \cdot h_s)^0 \right] ds \\ & - \frac{1}{4} \left(\overline{h_1 m_s + m_1 h_s}^2 + 3 \left(\overline{m_1^2 + h_1^2} \right) \right) + \overline{m_1 m_s} \cdot \overline{h_1 h_s} \end{aligned}$$

5.6. Sectional Curvature of Ω_0 . Now let $\zeta \in \Omega_0$. Since Ω_0 is a submanifold of Ω , its sectional curvature may be calculated using the Gauss Lemma.

5.6.1. *The Normal Bundle.* Let $z \in \tilde{\Omega}$.

$$Q = Q_1 + iQ_2 = \int_0^{2\pi} e^{z_1 + iz_2} dx$$

$z \in \tilde{\Omega}_0$ if and only if $Q = 0$. Now assume that $z \in \tilde{\Omega}_0$ such that $\chi(z) = \zeta$. Let $h \in T_z^0 \tilde{\Omega}$.

$$\begin{aligned} D_h Q &= \int_0^{2\pi} e^{z_1 + iz_2} (h_1 + ih_2) dx \\ &= \int_{\Delta} h e^{iz_2} ds \end{aligned}$$

Let

$$u_1 = (\cos z_2, -\sin z_2), u_2 = (\sin z_2, \cos z_2)$$

Then, in vector notation,

$$\begin{aligned} D_h Q &= \int_{\Delta} (h \cdot u_1, h \cdot u_2) ds \\ &= - \int_{\Delta} (h_s \cdot (D_s^{-2} u_1)_s, h_s \cdot (D_s^{-2} u_2)_s) ds \end{aligned}$$

Therefore, the gradient $\nabla Q_i = -D_s^{-2} u_i \in T_z^0 \tilde{\Omega}$. As a complex valued function, $D_s^{-1} u_1 = (D_s^{-1} e^{iz_2})^* = (c_z^0)^*$ where the superscript $*$ indicates complex conjugation. c_z^0 is the curve c_z translated so that its center of gravity is at the origin. Similarly, $D_s^{-1} u_2 = i (c_z^0)^*$ which is just a rotation of $(c_z^0)^*$ by $\pi/2$.

Clearly, $\nabla Q_1, \nabla Q_2$ are orthogonal. The normal vector space defined by the gradients ∇Q_1 and ∇Q_2 is invariant under change in z_2 by an additive constant and hence defines the normal vector space at $\zeta \in \Omega_0$. A choice of z corresponds to a choice of a basis of the normal vector space at ζ . For $i = 1, 2$,

$$\|\nabla Q_i\|^2 = \langle \nabla Q_i, \nabla Q_i \rangle = \int_{\Delta} |c_z^0|^2 ds$$

$\|\nabla Q_i\|^2$ is the polar moment of the curve c_z^0 , invariant under rotation of c_z^0 and hence, invariant under change in z_2 by an additive constant.

5.6.2. *The Second Fundamental Form.* The second fundamental form $S(m, h)$ has the decomposition:

$$S(m, h) = \frac{\langle S(m, h), \nabla Q_1 \rangle \nabla Q_1}{\|\nabla Q_1\|^2} + \frac{\langle S(m, h), \nabla Q_2 \rangle \nabla Q_2}{\|\nabla Q_2\|^2}$$

$$\langle S(m, h), \nabla Q_i \rangle = -\langle D_m \nabla Q_i, h \rangle = -\nabla^2 Q_i(m, h)$$

$$\text{The Hessian } \nabla^2 Q_i(m, h) = D_m D_h Q_i - D_{\Gamma(m, h)} Q_i$$

$$\begin{aligned} D_m D_h Q &= D_m \int_0^{2\pi} h e^z dx \\ &= \overline{m_1 h} \int_0^{2\pi} e^z dx + \int_0^{2\pi} m h e^z dx \\ &= \int_{\Delta} m h e^{iz_2} ds \text{ at } z \in \widetilde{\Omega}_0 \end{aligned}$$

since, at $z \in \widetilde{\Omega}_0$, $\int_0^{2\pi} e^z dx = 0$. $D_{\Gamma(m, h)} Q = \int_{\Delta} \Gamma(m, h) e^{iz_2} ds$. Therefore,

$$\langle S(m, h), \nabla Q \rangle = - \int_{\Delta} (mh - \Gamma(m, h)) e^{iz_2} ds$$

$$\langle S(m, h), \nabla Q_i \rangle = \langle \Gamma(m, h) - (mh), \nabla Q_i \rangle$$

Therefore,

$$\begin{aligned} S(m, h) &= \Gamma(m, h) - (mh) \\ \Gamma(m, h) &= -\frac{1}{2} D_s^{-1} \left[(m_1 h_s)^0 + (h_1 m_s)^0 + \left(D_s^{-1} (m_s \cdot h_s)^0, 0 \right) \right] \\ (mh) &= (m_1 h_1 - m_2 h_2, m_1 h_2 + m_2 h_1) \end{aligned}$$

5.6.3. *Gauss Lemma.*

$$\begin{aligned} &\kappa_{\zeta, \Omega_0}(m, h) \\ &= \kappa_{\zeta, \Omega} + \langle S(m, m), S(h, h) \rangle - \langle S(m, h), S(m, h) \rangle \\ &= \kappa_{\zeta, \Omega} + \langle \Gamma(m, m) - (m^2), \Gamma(h, h) - (h^2) \rangle \\ &\quad - \langle \Gamma(m, h) - (mh), \Gamma(m, h) - (mh) \rangle \end{aligned}$$

This may be simplified to take the form

$$\begin{aligned} \kappa_{\zeta, \Omega_0}(m, h) &= -3 \|h_1 m_s - m_1 h_s\|_0^2 - \frac{1}{2} \|h_2 m_s - m_2 h_s\|_0^2 \\ &\quad - 9 \overline{(h_1 m_2 - m_1 h_2)} (h_{1s} m_{2s} - m_{1s} h_{2s}) \\ &\quad - \frac{1}{2} (\overline{m_1^2} + \overline{h_1^2}) \end{aligned}$$

5.7. Sectional Curvature of $\Omega_0/\text{Diff}^+(S^1)$. Let B denote $\Omega_0/\text{Diff}^+(S^1)$. Let $\zeta \in \Omega_0$. Let $z \in \widetilde{\Omega}_0$ map to ζ . Let a, b be a pair of orthonormal tangent vectors at $\Pi(\zeta) \in B$. Let m, h be orthonormal Π -horizontal lifts of a, b in $T_z^0\widetilde{\Omega}_0$. Let $m^\#, h^\#$ be sections of $T^0\widetilde{\Omega}_0$ in a neighborhood of z which are Π -horizontal extensions of m, h . Then, by O'Neill's formula [7], we have

$$\kappa_{\Pi(\zeta), B}(a, b) = \kappa_{\zeta, \Omega_0}(m, h) + \frac{3}{4} \left\| \left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega_0}^V \right\|^2$$

where $\left[m_\theta^\#, h_\theta^\# \right]$ is the Lie bracket and as before, the superscript V denotes its Π -vertical component. We now derive an explicit expression for $\left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega_0}^V$.

We may regard the vector fields $m^\#, h^\#$ as vector fields on $\widetilde{\Omega}$ since $\left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega_0}^V = \left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega}$. We now construct the vector fields $m^\#, h^\#$. An important fact implicit in O'Neill's formula is that $\left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega}^V$ is independent of the choice of the horizontal extensions $m^\#, h^\#$. Extend the vectors m, h as sections of $T^0\widetilde{\Omega}$ such that m_θ, h_θ are constant vector fields on Ω and denote them again as m, h respectively. Set $m^\# = m^H$ and $h^\# = h^H$. Since $D_{m_\theta} h_\theta = D_{h_\theta} m_\theta = 0$, $\left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega_0}^V = D_{m_\theta} h_\theta^H - D_{h_\theta} m_\theta^H = D_{h_\theta} m_\theta^V - D_{m_\theta} h_\theta^V$. At the point ζ ,

$$\begin{aligned} D_{\zeta, h_\theta} m_\theta^V &= - (D_{\zeta, h} (L_1 L^{-1} (L_1^* \cdot m_{ss})))_\theta \\ &= - (L_1 L^{-1} D_{\zeta, h} (L_1^* \cdot m_{ss}))_\theta \end{aligned}$$

since $L_1^* \cdot m_{ss} = 0$ at ζ . Therefore,

$$\begin{aligned} \left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega_0}^V &= (L_1 L^{-1} \psi)_\theta \text{ where} \\ \psi &= D_{\zeta, m} (L_1^* \cdot h_{ss}) - D_{\zeta, h} (L_1^* \cdot m_{ss}) \end{aligned}$$

We use the following formulae:

$$\begin{aligned} D_{\zeta, h} m_s &= D_{\zeta, h} e^{-z_1} m_\theta = -h_1 m_s \\ D_{\zeta, h} m_{ss} &= -2h_1 m_{ss} - h_{1s} m_s \\ D_{\zeta, h} \kappa &= D_{\zeta, h} z_{2s} \\ &= -h_1 z_{2s} + h_{2s} \\ &= -h_1 \kappa + h_{2s} \\ D_{\zeta, h} L_1^* &= -h_1 L_1^* + (0, h_{2s}) \end{aligned}$$

We recall that κ without subscripts denotes the curvature of c_z and equals z_{2s} . Since $L_1^* \cdot m_{ss} = 0$ at ζ ,

$$\begin{aligned} D_{\zeta, h} (L_1^* \cdot m_{ss}) &= (D_{\zeta, h} L_1^*) \cdot m_{ss} + L_1^* \cdot D_{\zeta, h} m_{ss} \\ &= h_{2s} m_{2ss} - 2L_1^* \cdot (h_1 m_{ss}) - L_1^* \cdot (h_{1s} m_s) \\ &= h_{2s} m_{2ss} + 2h_{1s} m_{1ss} + (h_{1s} m_{1s})_s - \kappa (h_{1s} m_{2s}) \end{aligned}$$

taking into account again that $L_1^* \cdot m_{ss} = 0$ at z . We get

$$\begin{aligned} \psi &= (m_{2s}h_{2ss} - h_{2s}m_{2ss}) \\ &\quad + 2(m_{1s}h_{1ss} - h_{1s}m_{1ss}) + \kappa(h_{1s}m_{2s} - m_{1s}h_{2s}) \\ &= \left\| \left[m_\theta^\#, h_\theta^\# \right]_{\zeta, \Omega_0}^V \right\|^2 = \int_{\Delta} (L_1 L^{-1} \psi)_s \cdot (L_1 L^{-1} \psi)_s ds \\ &= \int_{\Delta} (L^{-1} \psi) [(L^{-1} \psi)_{ssss} - \kappa (\kappa L^{-1} \psi)_{ss}] ds = \int_{\Delta} \psi L^{-1} \psi ds \end{aligned}$$

5.8. An Absolute Bound.

5.8.1. *Sobolev Spaces.* Let $C^{\infty,0}(\Delta, \mathbb{R}) = \{f \in C^\infty(\Delta, \mathbb{R}) : \bar{f} = 0\}$. Define norms $\|f\|_n$, $n \in \mathbb{Z}$, by setting

$$\|f\|_n = \left[\int_{\Delta} |D_s^n f|^2 ds \right]^{\frac{1}{2}}$$

Let $H^n(\Delta, \mathbb{R})$ denote the completion of $C^{\infty,0}(\Delta, \mathbb{R})$ in the norm $\|\cdot\|_n$. If $(f_1, f_2) \in C^{\infty,0}(\Delta, \mathbb{R}^2)$, its norm

$$\|(f_1, f_2)\|_n = \left[\int_{\Delta} (|D_s^n f_1|^2 + |D_s^n f_2|^2) ds \right]^{\frac{1}{2}}$$

defines the corresponding space $H^n(\Delta, \mathbb{R}^2)$. If $f \in H^n(\Delta, \mathbb{R})$ and $g \in H^{-n}(\Delta, \mathbb{R})$, $fg \in L^1(\Delta, \mathbb{R})$:

$$\int_{\Delta} |fg| ds \leq \|f\|_n \|g\|_{-n}$$

The following estimate may be proved by means of the Fourier series of $f \in H^n(\Delta, \mathbb{R})$:

$$\|f\|_m \leq \frac{\|f\|_n}{(2\pi)^{n-m}}, \quad m < n$$

For $n > 0$, $H^n(\Delta, \mathbb{R})$ is an algebra in which the multiplication is defined as $(fg)^0$.

$$\begin{aligned} \|(fg)^0\|_1 &\leq \sqrt{2} \|f\|_1 \|g\|_1 \\ \|(fg)^0\|_n &\leq \frac{4\sqrt{3} \|f\|_n \|g\|_n}{\pi^{n-2}} \text{ if } n > 1 \end{aligned}$$

In what follows, we will refer to the function spaces defined above simply as H^n , L^1 , etc. and omit the specification (Δ, \mathbb{R}) or (Δ, \mathbb{R}^2) .

5.8.2. *An upper bound for $|\kappa_{\zeta, \Omega}|$.* We estimate below typical terms in the expression for $\kappa_{\zeta, \Omega}$ given above. If $f \in L^1$, $|\int_0^s f^0 ds| \leq |\int_0^s f ds| + |s\bar{f}|$ for $-\frac{1}{2} \leq s \leq \frac{1}{2}$. It follows that $|D_s^{-1}(f^0)| \leq \frac{3}{2} \|f\|_{L^1}$ and hence, $\|f^0\|_{-1} \leq \frac{3}{2} \|f\|_{L^1}$. Since m and h are unit vectors,

$$\begin{aligned} \left| D_s^{-1}(m_s \cdot h_s)^0 \right|^2 &\leq 9 \\ \left| D_s^{-1}(m_s \cdot m_s)^0 D_s^{-1}(h_s \cdot h_s)^0 \right| &\leq 9 \\ \overline{m_1^2} = \|m_1\|_0^2 &\leq \left(\frac{\|m_1\|_1}{2\pi} \right)^2 \leq \frac{1}{4\pi^2} \end{aligned}$$

$$\overline{|h_1 m_{1s}|} \leq \|h_1\|_0 (\|m_1\|_1 + \|m_1\|_1) \leq \frac{1}{\pi}$$

Substituting these inequalities in the expression for $\kappa_{\zeta, \Omega}$, we get

$$|\kappa_{\zeta, \Omega}| \leq \frac{9}{2} + \frac{19}{8\pi^2}$$

5.8.3. *An upper bound for $|\kappa_{\zeta, \Omega_0}|$.* $\|h_1 m_s\|_0 \leq \|h_1\|_{L^\infty} \|m_s\|_0 \leq \frac{3}{2}$. $\|h_1 m_s - m_1 h_s\|_0 \leq 3$.

$$|h_1 m_2 - m_1 h_2| \leq \frac{9}{2}.$$

$$\begin{aligned} & \left| \overline{(h_1 m_2 - m_1 h_2)(h_{1s} m_{2s} - m_{1s} h_{2s})} \right| \\ & \leq \frac{9}{2} \left| \overline{(h_{1s} m_{2s} - m_{1s} h_{2s})} \right| \leq \frac{9}{2} \overline{|h_s| |m_s|} \\ & \leq \frac{9}{2} (\|h_s\|_0 \|m_s\|_0) \leq \frac{9}{2} \end{aligned}$$

$$\begin{aligned} |\kappa_{\zeta, \Omega_0}| & \leq 3 \cdot 9 + \frac{9}{2} + 9 \cdot \frac{9}{2} + \frac{1}{2\pi^2} \\ & \leq 72 + \frac{1}{2\pi^2} \end{aligned}$$

5.8.4. *An upper bound for $|\kappa_{\Pi(\zeta), B}|$.* We estimate O'Neill's bracket.

$$\begin{aligned} \overline{\psi L^{-1} \psi} & = \int_{\Delta} (L^{-1} \psi) L (L^{-1} \psi) ds \\ & = \int_{\Delta} \left[(L^{-1} \psi)_{ss}^2 + (\kappa L^{-1} \psi)_s^2 \right] ds \end{aligned}$$

For $\overline{\psi L^{-1} \psi}$ to be finite, $L^{-1} \psi \in H^2$ or $\psi \in H^{-2}$. Orthonormality of m, h implies that $h_s, m_s \in H^0$ and $h_{ss}, m_{ss} \in H^{-1}$. This is not sufficient to place the terms like $m_{2s} h_{2ss}$ in H^{-2} . We have to assume additional regularity at least for one of the tangent vectors m, h . We assume that $h \in H^2$. We fix h and derive an upper bound with respect to m . There is an additional complication because $\overline{\psi}$ need not be equal to zero. We will see that under this assumption, $\psi \in L^1$. Therefore, $|\overline{\psi}| \leq \|\psi\|_{L^1}$ and $\|\psi^0\|_{-2} \leq \frac{1}{2\pi} \|\psi^0\|_{-1} \leq \frac{1}{2\pi} \cdot \frac{3}{2} \|\psi\|_{L^1}$.

Let $u = L^{-1} \psi$. $\overline{\psi L^{-1} \psi} = \overline{u \psi} = \overline{u L u}$. Now, $\overline{u \psi} = \overline{u} \overline{\psi} + \overline{u^0 \psi^0}$. Therefore,

$$\overline{\psi L^{-1} \psi} \leq |\overline{u}| |\overline{\psi}| + \|u^0\|_2 \|\psi^0\|_{-2}$$

Since $\overline{u L u} = \overline{u_{ss}^2} + (\kappa u)_s^2 \leq \overline{(u_{ss}^0)^2}$, $\|u^0\|_2 \leq \left(\overline{\psi L^{-1} \psi} \right)^{1/2}$. We get

$$\begin{aligned} \overline{\psi L^{-1} \psi} & \leq |\overline{u}| |\overline{\psi}| + \left(\overline{\psi L^{-1} \psi} \right)^{1/2} \|\psi^0\|_{-2} \\ & \leq |\overline{u}| \|\psi\|_{L^1} + \frac{3}{4\pi} \sqrt{\overline{\psi L^{-1} \psi}} \|\psi\|_{L^1} \\ \overline{\psi L^{-1} \psi} & \leq \left(\frac{3}{8\pi} + \sqrt{\left(\frac{3}{8\pi} \right)^2 + \frac{|\overline{u}|}{\|\psi\|_{L^1}}} \right)^2 \|\psi\|_{L^1}^2 \end{aligned}$$

We now proceed to estimate \overline{u} and $\|\psi\|_{L^1}$.

Let $L_* = D_s^4 - 4\pi^2 J^2 D_s^2$ where J is the rotation index. We have $\kappa = \kappa^0 + 2\pi J$. Let $M = L_0 - L = 2\pi J (D_s^2 \kappa^0 + \kappa^0 D_s^2) + \kappa^0 D_s^2 \kappa^0$. Let $v = L_0^{-1} \psi^0$ and let $w = L^{-1} \bar{\psi}$. Note that $4\pi^2 J^2 v = D_s^2 v - D_s^{-2} \psi^0$ so that $\bar{v} = 0$. Let $\psi_* = Mv$. Let $u_* = L^{-1} \psi_*$. We have

$$\begin{aligned} u &= w + v + u_* \\ \bar{u} &= \bar{w} + \bar{u}_* \end{aligned}$$

Let λ be the smallest eigenvalue of L .

$$\lambda \|w\|_0^2 \leq \overline{wLw} \leq |\bar{w}| |\bar{\psi}| \leq |\bar{\psi}| \|w\|_0$$

$$\text{Therefore, } \|w\|_0 \leq \frac{|\bar{\psi}|}{\lambda}$$

Note that the norms $\|\cdot\|$ extend to $C^\infty(\Delta, \mathbb{R})$ if $n \geq 0$. Since $|\bar{w}| \leq \|w\|_0$,

$$|\bar{w}| \leq \frac{|\bar{\psi}|}{\lambda}$$

Since $\psi \in H^{-2}$, $\psi^0 \in H^{-2}$, $u, v \in H^2$, $\psi_* \in H^0$ and $u_* \in H^4$.

$$\begin{aligned} \lambda \|u_*\|_0^2 &\leq \overline{u_* L u_*} \leq \overline{u_* \psi_*} \leq \|u_*\|_0 \|\psi_*\|_0 \\ |\bar{u}_*| &\leq \|u_*\|_0 \leq \frac{\|\psi_*\|_0}{\lambda} \end{aligned}$$

It follows that

$$|\bar{u}| \leq \frac{1}{\lambda} (|\bar{\psi}| + \|\psi_*\|_0)$$

Let

$$\begin{aligned} D_s^{-2} \psi^0 &= \sum_{k \neq 0} a_k e^{2\pi i k s} \\ v &= \sum_{k \neq 0} b_k e^{2\pi i k s} \end{aligned}$$

Substituting these in the equation $v_{ss} - 4\pi^2 J^2 v = D_s^{-2} \psi^0$, we get

$$\begin{aligned} b_k &= -\frac{a_k}{4\pi^2(k^2 + J^2)} \\ v_{ss} &= \sum_{k \neq 0} \frac{k^2}{k^2 + J^2} a_k e^{2\pi i k s} \end{aligned}$$

$$\begin{aligned} \|v\|_2 &= \|v_{ss}\|_0 = \left[\sum_{k \neq 0} \left(\frac{k^2}{k^2 + J^2} a_k \right)^2 \right]^{1/2} \\ &\leq \left[\sum_{k \neq 0} a_k^2 \right]^{1/2} \leq \|\psi^0\|_{-2} \end{aligned}$$

$$\psi_* = 2\pi J ((\kappa^0 v)_{ss} + \kappa^0 v_{ss}) + \kappa^0 (\kappa^0 v)_{ss}.$$

$$\begin{aligned} \|(\kappa^0 v)_{ss}\|_0 &= \|\kappa^0 v\|_2 \leq 4\sqrt{3} \|\kappa^0\|_2 \|v\|_2 \\ &\leq 4\sqrt{3} \|\kappa\|_2 \|\psi^0\|_{-2} \end{aligned}$$

$$\|\kappa^0 v_{ss}\|_0 \leq \|\kappa\|_{L^\infty} \|\psi^0\|_{-2}$$

$$\begin{aligned} \|\kappa^0 (\kappa^0 v)_{ss}\|_0 &\leq \|\kappa\|_{L^\infty} \|(\kappa^0 v)_{ss}\|_0 \\ &\leq 4\sqrt{3} \|\kappa\|_{L^\infty} \|\kappa\|_2 \|\psi^0\|_{-2} \end{aligned}$$

We also have $\|\psi^0\|_{-2} \leq \frac{3}{4\pi} \|\psi\|_{L^1}$. We get

$$\|\psi_*\|_0 \leq \frac{3\sqrt{3}}{\pi} \|\kappa\|_{L^\infty} \|\kappa\|_2 \left(1 + \frac{2\pi J}{\|\kappa\|_{L^\infty}} + \frac{2\pi J}{4\sqrt{3} \|\kappa\|_2} \right) \|\psi\|_{L^1}$$

$$\begin{aligned} \psi &= (m_{2s} h_{2ss} - h_{2s} m_{2ss}) \\ &\quad + 2(m_{1s} h_{1ss} - h_{1s} m_{1ss}) + \kappa (h_{1s} m_{2s} - m_{1s} h_{2s}) \end{aligned}$$

Estimates for typical terms are as follows.

$$\begin{aligned} \|m_{2s} h_{2ss}\|_{L^1} &\leq \|m_{2s}\|_0 \|h_{2ss}\|_0 \leq \|h\|_2 \\ \|m_{2ss} h_{2s}\|_{L^1} &\leq \|m_{2ss}\|_{-1} \|h_{2s}\|_1 \leq \|h\|_2 \\ \|\kappa m_{1s} h_{2s}\|_{L^1} &\leq \|\kappa\|_{L^\infty} \|m_{1s}\|_0 \|h_{2s}\|_0 \leq \|\kappa\|_{L^\infty} \end{aligned}$$

Therefore,

$$\|\psi\|_{L^1} \leq 6 \|h\|_2 + 2 \|\kappa\|_{L^\infty}$$

5.8.5. *A Lower Bound for λ .* In this section, H^n denotes the completion of $C^\infty(\Delta, \mathbb{R})$ or $C^\infty(\Delta, \mathbb{R}^2)$ in the norm $\|\cdot\|_n$. Recall that $L_1 = (D_s, \kappa)$. Let A be the positive definite operator $-D_s^2 + \kappa^2$. For $b, b_1, b_2 \in H^2$, Let

$$\begin{aligned} \langle b_1, b_2 \rangle_A &= \int_\Delta b_1 A b_2 ds = \int_\Delta (L_1 b_1) \cdot (L_1 b_2) ds \\ \|b\|_A^2 &= \langle b, b \rangle_A = \int_\Delta (L_1 b) \cdot (L_1 b) ds = \|L_1 b\|_0^2 \\ \langle b_1, b_2 \rangle_L &= \int_\Delta b_1 L b_2 ds = \int_\Delta (L_1 b_1)_s \cdot (L_1 b_2)_s ds \\ \|b\|_L^2 &= \langle b, b \rangle_L = \int_\Delta (L_1 b)_s \cdot (L_1 b)_s ds = \|L_1 b\|_1^2 \end{aligned}$$

We have the inequality

$$\left\| (L_1 b)^0 \right\|_0^2 \leq \frac{\|L_1 b\|_1^2}{4\pi^2}$$

and the estimate [1],

$$\|(L_1 b)\|_0^2 = \|b\|_A^2 \geq \frac{1}{2} \|b\|_0^2$$

We need to relate $\left\| (L_1 b)^0 \right\|_0$ and $\|L_1 b\|_0$.

Let $U = (0, 1) \in T_z \tilde{\Omega}$. Then, $L_1 b = (L_1 b)^0 + \overline{\kappa} b U$. $L_1 b = (L_1 b)^0$ if and only if

$$0 = \int_\Delta \kappa b ds = \int_\Delta b A (A^{-1} \kappa) ds = \langle b, A^{-1} \kappa \rangle_A$$

Let $\beta = A^{-1} \kappa$. $\overline{\kappa \beta} = \|\beta\|_A^2 = \overline{(\beta_s^2 + \kappa^2 \beta^2)} \geq \|\kappa \beta\|_0^2$. We also have $\overline{\kappa \beta} \leq \|\kappa \beta\|_0$ by Schwartz inequality. Therefore, $\overline{\kappa \beta} \leq (\overline{\kappa \beta})^{1/2}$ and hence, $\|\beta\|_A \leq 1$. Equality

requires that β is constant which, in turn, implies that κ is constant, that is, c_z is a circle. Write $b = b_0 + a\beta$, $a = \frac{\langle b, \beta \rangle_A}{\|\beta\|_A^2}$.

$$\|b\|_A^2 = \|b_0\|_A^2 + a^2 \|\beta\|_A^2$$

$$L_1 b = (L_1 \beta)^0 + \overline{\kappa \beta} U = (L_1 \beta)^0 + \|\beta\|_A^2 U. \quad (L_1 b)^0 = (L_1 b_0) + a (L_1 \beta)^0 = (L_1 b_0) + a L_1 \beta - a \|\beta\|_A^2 U.$$

$$\|(L_1 b)^0\|_0^2 = \|b_0\|_A^2 + a^2 \|\beta\|_A^2 (1 - \|\beta\|_A^2)$$

$$\frac{\|(L_1 b)^0\|_0^2}{\|(L_1 b)\|_0^2} = \frac{\|b_0\|_A^2 + a^2 \|\beta\|_A^2 (1 - \|\beta\|_A^2)}{\|b_0\|_A^2 + a^2 \|\beta\|_A^2} \geq 1 - \|\beta\|_A^2$$

$$\begin{aligned} \frac{\|b\|_L^2}{\|b\|_0^2} &\geq 4\pi^2 \frac{\|(L_1 b)^0\|_0^2}{\|b\|_0^2} \\ &\geq 4\pi^2 \frac{\|b\|_A^2}{\|b\|_0^2} \geq 2\pi^2 \end{aligned}$$

Therefore,

$$\lambda = \min_{b \neq 0} \frac{\|b\|_L^2}{\|b\|_0^2} \geq 2\pi^2 (1 - \overline{\kappa A^{-1} \kappa})$$

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