

AN H^2 RIEMANNIAN METRIC ON THE SPACE OF PLANAR CURVES MODULO SIMILITUDES

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ABSTRACT. Analyzing shape manifolds as Riemannian manifolds has been shown to be an effective technique for understanding their geometry. Riemannian metrics of the type H^0 and H^1 on the space of planar curves have already been investigated in detail. Here, an H^2 metric is formulated which in particular depends on the local bending of the curve rather than changes in its local orientation. Since in many applications, the basic shape of an object is understood to be independent of its scale, orientation or placement, we consider the space of planar curves modulo similitudes. Equations of the geodesic for parametrized curves as well as for unparametrized curves have been derived. Equations of gradient descent are given for constructing the geodesics between two given curves numerically.

1. INTRODUCTION

One of the approaches to shape analysis is to study the Riemannian geometry of the shape manifold. Examples of this approach may be seen in [1-13]. In particular, there has been a great deal of progress in understanding the geometry of planar shapes exemplified by silhouettes and MRI. Mathematically, this space is the space of smooth planar curves. Michor and Mumford have analyzed this space in considerable detail. In particular, they derive the geodesic equation for a general Sobolev metric. A more detailed analysis including the sectional curvature of specific metrics is carried out in [6,13]. Conformal versions of the H^0 -metric of Michor and Mumford are studied in [6] while an H^1 -metric is studied in [13]. An H^2 -metric on space of curves parametrized by the arc length has been formulated in [1].

It is useful to separate out the pose and scale of an object from its inherent geometry. This leads us to study the space of planar curves modulo translation, rotation and scale. An example of such a formulation is an H^1 -metric analyzed in [13] by Younes et al. Although rotations have been factored out, the metric still depends on local rotations of the curve. To obtain a metric which depends purely on the bending and stretching of the curve, what is needed is an H^2 -metric. In this paper, we construct such a metric and derive the geodesic equation. We also give the equation of gradient descent for deforming a given curve into a geodesic.

2. SOBOLEV METRICS

Let $\text{Imm}(S^1, \mathbb{R}^2)$ denote the space of all C^∞ immersions $c : S^1 \rightarrow \mathbb{R}^2$. The unit circle S^1 is parametrized by θ . A tangent vector h at a point c in $\text{Imm}(S^1, \mathbb{R}^2)$ is just a vector field in \mathbb{R}^2 along the image curve of c in \mathbb{R}^2 . Let v and n denote the vector fields of unit tangent and unit normal vectors along c and let s denote

the arc-length. The infinitesimal arc-length $ds = |c_\theta| d\theta$. (A subscript denotes the derivative.) As vector fields along c viewed as a curve in the complex plane \mathbb{C} , $v = c_s$ and $n = ic_s$. If a, b are vectors in \mathbb{R}^2 and $a \cdot b$ will denote their dot product. We note that the complex vector $h_s/c_s = h_s \cdot v + ih_s \cdot n$

Let Σ denote the group of similitudes. Its action on c as curve in \mathbb{C} is given by $\alpha c + \beta$ where α, β are (complex) constants. The vertical vectors at c of the quotient map $\rho : \text{Imm}(S^1, \mathbb{R}^2) \rightarrow \text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ have the form $\alpha c + \beta$. If h is a vertical vector, then h_s/c_s is constant, that is, $h_s \cdot v$ and $h_s \cdot n$ are constant. We define Σ -horizontal vectors at c as

$$\{h | \bar{h} = \overline{h_s \cdot v} = \overline{h_s \cdot n} = 0\}$$

where the bar over a variable denotes its average value: $\bar{f} = \frac{1}{\ell} \int_c f ds$, $\ell = \int_c ds$ = the length of c . The horizontal and the vertical vectors define decomposition of the tangent bundle of $\text{Imm}(S^1, \mathbb{R}^2)$ into two subbundles such that the tangent map $T_c \text{Imm}(S^1, \mathbb{R}^2) \rightarrow T_{\rho(c)} \text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ is an isomorphism when restricted to the horizontal vectors. We identify the tangent vectors at $\rho(c)$ with the horizontal tangent vectors at c . The projection from the tangent space at c onto the space of horizontal tangent vectors is given by the formula

$$h \mapsto h^\Sigma = (h - \bar{h}) - \overline{h_s/c_s}(c - \bar{c})$$

Given a path $c(t)$ in $\text{Imm}(S^1, \mathbb{R}^2)$, there exists a horizontal path $c^\Sigma(t) = \alpha(t)c(t) + \beta(t)$ where $\alpha(t)$ and $\beta(t)$ are given by equations

$$\begin{aligned} \alpha_t + \overline{\alpha c_{ts}/c_s} &= 0 \\ \beta_t + \overline{(\alpha c)_t} &= 0 \end{aligned}$$

such that $c(t)$ and $c^\Sigma(t)$ project to the same path in $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$.

We now define a bilinear form on the tangent space at c :

$$\langle m, h \rangle = \ell^{2n-3} \int_c (D_s^{n-1}(m_s^\Sigma \cdot v), D_s^{n-1}(m_s^\Sigma \cdot n)) \cdot (D_s^{n-1}(h_s^\Sigma \cdot v), D_s^{n-1}(h_s^\Sigma \cdot n)) ds, \quad n \geq 1$$

where $D_s = d/ds$ and ℓ is the length of c . Note that $h_s^\Sigma \cdot v = h_s \cdot v - \overline{h_s \cdot v}$ and $h_s^\Sigma \cdot n = h_s \cdot n - \overline{h_s \cdot n}$. Moreover $\langle h, h \rangle = 0$ if and only if h is a vertical vector. $\langle m, h \rangle$ is non-degenerate on the horizontal subbundle and defines a Riemannian metric on it and hence on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. The case $n = 1$ has been treated in [13] using a representation of $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ by grassmannians. Here we consider the case $n = 2$. Since $D_s^{n-1}(h_s^\Sigma \cdot v, h_s^\Sigma \cdot n) = D_s^{n-1}(h_s \cdot v, h_s \cdot n)$ if $n \geq 2$,

$$\langle m, h \rangle = \ell^{2n-3} \int_c (D_s^{n-1}(m_s \cdot v, m_s \cdot n) \cdot (D_s^{n-1}(h_s \cdot v, h_s \cdot n)) ds, \quad n \geq 2$$

Given a path $c(t) : [0, 1] \rightarrow \text{Imm}(S^1, \mathbb{R}^2)$, we define its length by setting

$$L(c(t)) = \int_0^1 \sqrt{\langle c_t, c_t \rangle} dt$$

The length of a path is invariant under the action of Σ and equals zero if and only if the path is vertical.

3. GEODESIC EQUATION FOR THE H^2 -METRIC ON $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$

Let $c(t) : [0, 1] \rightarrow \text{Imm}(S^1, \mathbb{R}^2)$ be a lift of a path in $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. We may assume that for each t , the curve $c(t)$ has unit length. Deformation of a curve by a horizontal tangent vector preserves its length. We get the equation of a geodesic by calculating the first variation of the energy

$$E(c) = \frac{1}{2} \int_0^1 \langle c_t, c_t \rangle dt$$

Let $m(t)$ be a field of horizontal tangent vectors along $c(t)$, vanishing at its endpoints. We have the following formulas:

$$\begin{aligned} D_m |c_\theta| &= (m_s \cdot v) |c_\theta|, & D_m D_s &= -(m_s \cdot v) D_s \\ D_m v &= (m_s \cdot n) n, & D_m n &= -(m_s \cdot n) v \\ D_m (c_{ts} \cdot v) &= (m_s \cdot v)_t, & D_m (c_{ts} \cdot n) &= (m_s \cdot n)_t \end{aligned}$$

$$\begin{aligned} D_m E(c) &= \frac{1}{2} \int_0^1 D_m \int_{S^1} ((c_{ts} \cdot v)_\theta^2 + (c_{ts} \cdot n)_\theta^2) \frac{d\theta}{|c_\theta|} dt \\ &= \int_0^1 \int_{S^1} (m_s \cdot v, m_s \cdot n)_{\theta t} \cdot (c_{ts} \cdot v, c_{ts} \cdot n)_\theta \frac{d\theta}{|c_\theta|} dt \\ &\quad - \frac{1}{2} \int_0^1 \int_{S^1} (m_s \cdot v) ((c_{ts} \cdot v)_\theta^2 + (c_{ts} \cdot n)_\theta^2) \frac{d\theta}{|c_\theta|} dt \\ &= - \int_0^1 \int_c (m_s \cdot v, m_s \cdot n)_s \cdot (c_{ts} \cdot v, c_{ts} \cdot n)_{st} ds dt \\ &\quad + \frac{1}{2} \int_0^1 \int_c (m_s \cdot v)_s D_s^{-1} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 ds dt \\ &= - \int_0^1 \langle m, \gamma(c) \rangle dt \end{aligned}$$

Here, the superscript 0 on a quantity means the quantity minus its average value: $f^0 = f - \bar{f}$. The operator D_s^{-1} is uniquely defined by requiring that $D_s^{-1} f = 0$; $D_s^{-1} f$ may be calculated efficiently by means of FFT. The symbol γ denotes the geodesic curvature of the path $c(t)$. An explicit expression for γ may be derived as follows.

$$\begin{aligned} (\gamma_s \cdot v)_s &= (c_{ts} \cdot v)_{st} - \frac{1}{2} D_s^{-1} ((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2)^0 \\ (\gamma_s \cdot n)_s &= (c_{ts} \cdot n)_{st} \\ (c_{ts} \cdot v)_{st} &= (c_{ts} \cdot v)_{ts} - (c_{ts} \cdot v)(c_{ts} \cdot v)_s \\ &= (c_{tts} \cdot v - (c_{ts} \cdot v)^2 + (c_{ts} \cdot n)^2)_s - (c_{ts} \cdot v)(c_{ts} \cdot v)_s \\ &= \left(c_{tts} \cdot v - \frac{3}{2} (c_{ts} \cdot v)^2 + (c_{ts} \cdot n)^2 \right)_s \\ (c_{ts} \cdot n)_{st} &= (c_{ts} \cdot n)_{ts} - (c_{ts} \cdot v)(c_{ts} \cdot n)_s \\ &= (c_{tts} \cdot n - 2(c_{ts} \cdot v)(c_{ts} \cdot n))_s - (c_{ts} \cdot v)(c_{ts} \cdot n)_s \end{aligned}$$

Let Λ be the complex vector

$$\Lambda = \left[\left(\frac{3}{2}(c_{ts} \cdot v)^2 + (c_{ts} \cdot n)^2 \right)^0 - \frac{1}{2} D_s^{-2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 \right] \\ + i \left[\left(2(c_{ts} \cdot v)(c_{ts} \cdot n) \right)^0 + D_s^{-1} \left((c_{ts} \cdot v)(c_{ts} \cdot n)_s \right)^0 \right]$$

Since $\int_c m_s ds = 0$, we have

$$0 = \alpha \int_c (m_s/c_s)^* c_s^* ds = \alpha \int_c (m_s/c_s)_s^* D_s^{-1}(c_s^*) ds \text{ for all } \alpha \in \mathbb{C}$$

where the superscript $*$ denotes the complex conjugate.

$$D_m E(c) = -\operatorname{Re} \int_0^1 \int_c (m_s/c_s)_s^* \left[(c_{tts}/c_s) - \Lambda - \alpha D_s^{-2}(c_s^*) \right]_s ds$$

Therefore,

$$\gamma_s/c_s = (c_{tts}/c_s) - \Lambda - \alpha D_s^{-2}(c_s^*)$$

Since we must have $\int_c \gamma_s ds = 0$, set

$$\alpha = \frac{\int_c \Lambda c_s ds}{\int_c |D_s^{-1}(c_s)|^2 ds}$$

Finally,

$$\gamma = c_{tt} - D_s^{-1} \left((\Lambda - \alpha D_s^{-2}(c_s^*) c_s \right)$$

A path $c(t)$ is a geodesic if and only if $\gamma(c) = 0$. A path $c(t)$ may be deformed into a geodesic by gradient descent: $\frac{\partial c}{\partial \tau} \propto \gamma$.

An alternative form of the geodesic equation may be obtained by integration by parts:

$$\langle m, \gamma \rangle = \int_c m \cdot \left((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n \right)_s ds$$

for $m \in T_c \operatorname{Imm}(S^1, \mathbb{R}^2)$. The geodesic equation may be written as

$$\left((c_{ts} \cdot v)_{sts} v + (c_{ts} \cdot n)_{sts} n - \frac{1}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 v \right)_s = 0$$

This equation may be more conveniently written in terms of the "momentum" vector u :

$$u = \left((c_{ts} \cdot v)_{ss} v + (c_{ts} \cdot n)_{ss} n \right)_s$$

As a vector field in \mathbb{C} , $u = \left((c_{ts}/c_s)_{ss} c_s \right)_s$.

$$u_t = \left(-2c_{ts} \cdot v + i c_{ts} \cdot n \right) u \\ + \left(-c_{ts} \cdot v + i c_{ts} \cdot n \right)_s \left(c_{ts}/c_s \right)_{ss} c_s + \left((c_{ts}/c_s)_{sts} c_s \right)_s$$

Substituting this in the geodesic equation

$$0 = \left((\gamma_s/c_s)_{ss} c_s \right)_s = \left((c_{ts}/c_s)_{sts} c_s \right)_s - \frac{1}{2} \left(\left(\| (c_{ts}/c_s)_s \|^2 \right)^0 c_s \right)_s$$

we get

$$\begin{aligned}
 & u_t + (2c_{ts} \cdot v - ic_{ts} \cdot n)u \\
 = & (-c_{ts} \cdot v + ic_{ts} \cdot n)_s (c_{ts}/c_s)_{ss} c_s \\
 & + \frac{1}{2} \left((\| (c_{ts}/c_s)_s \|^2)^0 c_s \right)_s \\
 = & i[(c_{ts} \cdot v)_{ss} (c_{ts} \cdot n)_s - (c_{ts} \cdot v)_s (c_{ts} \cdot n)_{ss}] \\
 & + \frac{\kappa}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 c_s
 \end{aligned}$$

Since $\langle ((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s^\Sigma, \gamma \rangle = \langle ((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s, \gamma \rangle \geq 0$, the gradient descent equation takes form

$$\frac{\partial c}{\partial \tau} \propto \left((c_{ts} \cdot v)_{sts} v + (c_{ts} \cdot n)_{sts} n - \frac{1}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 v \right)_s^\Sigma$$

4. GEODESIC EQUATION FOR THE H^2 -METRIC ON $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$

The group of diffeomorphisms $\text{Diff}(S^1)$ acts on $\text{Imm}(S^1, \mathbb{R}^2)$ by composition from the right. The action commutes with the action of Σ and hence $\text{Diff}(S^1)$ acts on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. The quotient $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ is the space of unparametrized curves modulo similitudes which inherits a Riemannian metric making the quotient map a submersion. The tangent bundle of $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ splits into a vertical and a horizontal subbundles which are orthogonal with respect to the metric on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. The geodesics on $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ are just the horizontal geodesics on $\text{Imm}(S^1, \mathbb{R}^2)/\Sigma$. If a Σ -horizontal path in $\text{Imm}(S^1, \mathbb{R}^2)$ is also horizontal with respect to $\text{Diff}(S^1)$, then its geodesic curvature γ is $\text{Diff}(S^1)$ -horizontal as well. The action of an infinitesimal diffeomorphism corresponds to a vector field of the form bv along c where $b \in \mathbb{R}$. A tangent vector $h \in T_{\rho(c)} \text{Imm}(S^1, \mathbb{R}^2)/\Sigma$ is horizontal

$$\begin{aligned}
 \iff & \langle bv, h \rangle = \int_c bv \cdot ((h_s \cdot v)_{ss} v + (h_s \cdot n)_{ss} n)_s ds = 0 \\
 \iff & v \cdot ((h_s \cdot v)_{ss} v + (h_s \cdot n)_{ss} n)_s = (h_s \cdot v)_{sss} - \kappa (h_s \cdot n)_{ss} = 0
 \end{aligned}$$

where $\kappa = v_s$ is the curvature of c . Therefore, the geodesic curvature γ satisfies the equation $((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s \cdot v = 0$ so that

$$((\gamma_s \cdot v)_{ss} v + (\gamma_s \cdot n)_{ss} n)_s = \left((c_{ts} \cdot v)_{sts} \kappa + (c_{ts} \cdot n)_{sts} - \frac{\kappa}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 \right) n$$

The geodesic equation takes the form

$$(c_{ts} \cdot v)_{sts} \kappa + (c_{ts} \cdot n)_{sts} = \frac{\kappa}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0$$

Now if a path $c(t)$ is horizontal with respect to $\text{Diff}(S^1)$, then $u \cdot v = 0$ so that $u = an$, $a(t) \in \mathbb{R}$. Therefore, in terms the momentum u , the geodesic equation takes the form

$$\begin{aligned}
 a_t + 2(c_{ts} \cdot v)_s a & = (c_{ts} \cdot v)_{ss} (c_{ts} \cdot n)_s - (c_{ts} \cdot v)_s (c_{ts} \cdot n)_{ss} \\
 & + \frac{\kappa}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0
 \end{aligned}$$

We may construct geodesics in $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$ as follows. Given a path $C(t)$ in $\text{Imm}(S^1, \mathbb{R}^2)/(\Sigma, \text{Diff}(S^1))$, lift it to a path in $\text{Imm}(S^1, \mathbb{R}^2)$ which is

horizontal with respect to Σ as well as $\text{Diff}(S^1)$ and then deform it by the gradient descent equation

$$\frac{\partial c}{\partial \tau} \propto \left((c_{ts} \cdot v)_{sts} \kappa + (c_{ts} \cdot n)_{stss} - \frac{\kappa}{2} \left((c_{ts} \cdot v)_s^2 + (c_{ts} \cdot n)_s^2 \right)^0 \right)^\Sigma$$

A horizontal path $c(t)$ may be constructed as follows. Lift $C(t)$ to a path $c(t)$ on $\text{Imm}(S^1, \mathbb{R}^2)$. We now construct a path $\varphi(t, \theta) \in \text{Diff}(S^1)$ such that $c \circ \varphi$ is horizontal. Write $\varphi_t(t, \theta) = \xi(t, \theta) \circ \varphi(t, \theta)$ and let $\eta = |c_\theta| \xi$. Note that $(c \circ \varphi)_t = (c_t + \eta v) \circ \varphi$ and $D_{c \circ \varphi}(f \circ \varphi) = (D_c f) \circ \varphi$ (see [3]). The path $c \circ \varphi$ is horizontal with respect to $\text{Diff}(S^1)$

$$\begin{aligned} \iff & ((c_t + \eta v)_s \cdot v)_{sss} - \kappa ((c_t + \eta v)_s \cdot n)_{ss} = 0 \\ \iff & L\eta = (c_{ts} \cdot v)_{sss} - \kappa (c_{ts} \cdot n)_{ss} \end{aligned}$$

where $L\eta = -\eta_{ssss} + \kappa(\kappa_c \eta)_{ss}$. The operator L is self-adjoint and it is positive definite provided that c is not a circle. Assuming that the path does not pass through a circle, η may be calculated by minimizing

$$\int_c \frac{1}{2} (\eta_{ss}^2 + (\kappa \eta)_s^2) + ((c_{ts} \cdot v)_{sss} - \kappa (c_{ts} \cdot n)_{ss}) \eta ds$$

by gradient descent. Then, integrate the equation $\varphi_t(t, \theta) = \xi(t, \theta) \circ \varphi(t, \theta)$ to obtain φ and the horizontal path $(c \circ \varphi)^\Sigma$.

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