Irreducible components of families of level algebras

Anthony Iarrobino
Northeastern University

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(IIIB: Joint with Mats Boij, KTH, Stockholm)

- I. LevAlg(H) - Intro
  - A. Macaulay duality and level algebras
  - B. Level Algebras when $\exists$ structure theorem
  - C. Why do we care?
  - D. Remark: generic level algebras
- II. Which Hilbert functions occur?
  - A. Recent progress
  - B. Zanello's non-unimodal H, r=3.
- III. LevAlg(H), reducible
  - A. For $r \geq 4, t = 1$.
  - B. For $r = 3$, types t=2,3, 4, ...
    Several infinite series of examples.
- IV. Overview, open problems.
  - A. What do we want to learn?
  - B. Deformations (see IC.)
  - C. Problem list.
- V. Annotated References (9p)

A goal of the talk is to give a personal view of open problems about Artinian level algebras.

I. **Level Algebras**: An Introduction

$R = k[x_1, \dots, x_r]$, polynom, $M = (x_1, \dots, x_r)$.
$A = R/I$ Artinian: $\dim_k A = n$.
$Soc(A) = 0 : M = \{a \in A \mid Ma = 0\}$.
Type $t(A) = \dim_k Soc(A)$.

Graded Artinian (GA) $A = A_0 \oplus A_1 \oplus \cdots \oplus A_j$
Level Algebra: Also $Soc(A) = A_j$.
Gorenstein GA: Also t(A)=1.

Sequence $H = (1, h_1, \dots, h_i, \dots, h_j)$.
Hilbert function $H = H(A) : \ h_i = H(A)_i = \dim_k A_i$

Def. $\mathrm{LevAlg}_R(t, j) = \{$level graded Artinian quotients A of R, socle degree j, type t.$\}$
$\mathrm{LevAlg}_R(H) = \{$level algs $A = R/I \mid H(A) = H.\}$

Def. Macaulay duality: R acts on $S = k[X_1, \dots, X_r]$ by PDO.

$$g \in R_i, F \in S_j,\ 0 \leq i \leq j,\ g \circ F \in S_{j-i}.$$

$$g \circ F = g(\partial/\partial X_1, \dots \partial/\partial X_r) \circ F.$$

Let $V \subset S_j$; then define ideal
$\mathrm{Ann}\, V = \{f \in R \mid f \circ V = 0\}$.
Let I ideal of R; then define
$I^{\perp} = \{F \in S \mid I \circ F = 0\}, R-$submodule of S.
${I_i}^{\perp} = [I^{\perp}]_i = \{F \in S_i \mid I \circ F = 0\}$, subspace of S_i.

IA. Macaulay duality and level algebras

Thm. (Macaulay): $\exists$ 1-1 correspondence,

$A \in \mathrm{LevAlg}_R(t,j) \leftrightarrow$ vector sp. $W \subset S_j$, dim W=t

$A = R/I \rightarrow W_A = {I_j}^{\perp} \subset S_j$.
$W \subset S_j \rightarrow A_W = R/AnnW \in$ LevAlg(t,j).

$\Rightarrow \mathrm{LevAlg}_R(t,j) \subset Grass(t, S_j)$.

Lemma 1. Assume $A = R/I \in \mathrm{LevAlg}_R(t,j)$. Then I is determined by I_j. For $i \leq j$
$I_i = I_j : R_{j-i}$ and
$I_i^{\perp} = R_{j-i} \circ V_A \cong \hat{A}_i = \mathrm{Hom}_k(A,k)_i$.

Corollary. $H(A)_i = dim_k(R_{j-i} \circ V_A)$.

Notation: $W \subset S_j$, then $I_W = AnnW$.

Ex 1. A Gorenstein.

i. $F = X^3 + Y^3 + Z^3$, $R = k[x,y,z], S = k[X,Y,Z]$.
$I_F = (xy, xz, yz, x^3 - y^3, x^3 - z^3)$.
$A_F = R/I_F, \quad H(A_F) = 1,3,3,1)$. “compressed”.
$\hat{A_F} = \langle 1; X, Y, Z; X^2, Y^2, Z^2; F\rangle$.

ii. $F = X^4 + Y^4 + Z^4 + (X + Y + Z)^4 \in S_4$
$I_F = (xy - xz, xy - yz, x^4 - y^4, x^4 - z^4, z^5)$
$H(A_F) = (1,3,4,3,1)$. (F sum of four powers).

iii. $F = X^4 + Y^4 + Z^4 + {L_1}^4 + {L_2}^4 + {L_3}^4$,
$L_1, L_2, L_3 \in R_1$ general enough.
$H_F = (1,3,6,3,1)$, A_F “compressed, extremal”.

Def. A "compressed": length $n(A)$ maximum given t, j, r.

Ex 2 A type $t \geq 2$.

i. $W = X^4 + Y^4 + Z^4, {L_1}^4 + {L_2}^4 + {L_3}^4$,
$L_i \in R_1$ general enough.
$H_F = (1, 3, 6, 6, 2)$ "compressed".

ii. $S = k[X, Y, Z], W = \langle ZX^2 + ZY^2, Z^2X, Z^2Y \rangle$,
$I_W = (xy, x^2 - y^2, z^3, M^4)$
$\hat{A} = \langle 1; X, Y, Z; X^2 + Y^2, ZX, ZY, Z^2; W \rangle$.
$H(A) = (1, 3, 4, 3)$.

Summary of Macaulay duality and level algebras:

- The higher partial derivatives of homog. forms in a vector space W, have additional structure as the dual $\hat{A_W}$ where $A_W = R/I, I = \text{Ann } W$.
- Plan: use algebra techniques such as minimal resolution to study A_W, determine $H(A_W)$, also dimension, closure, and irreducible components of $\text{LevAlg}_R(H)$

IB. Striking success given a structure theorem!

- CM height two (Hilbert-Burch):
  $I =$ maximal minors of $(n-1)\times n$ matrix.
- Gorenstein height three (Buchsbaum-Eisenbud):
  $I =$ Pfaffians of alternating matrix M.

• Gorenstein case $t = 1, r = 3$.

Hilbert functions (Buchsbaum-Eisenbud, Stanley)

- All Gorenstein H are *unimodal* for r=3,t=1.
- $\Delta H_{\leq j/2}$ is an O-sequence of height two.
  Here $\Delta H_i = h_i - h_{i-1}$.

Ex: $H = (1,3,6,8,10,11,11,10,8,6,3,1)$, $j = 11$
$\Delta H_{\leq j/2} = (1,2,3,2,2,1) = H(k[x,y]/(x^3,x^2y,y^5))$.

Parametrization (t=1, r=3):

- (Diesel) Gor(H) is irreducible!
- (Kleppe, Conca-Valla & Boij): Gor(H) is smooth
- (Boij) Betti strata and their closures in Gor(H)
- (Kanev-I) Closures of strata partially known

• CM Height two level algebras (r=2):

Hilbert functions: (Iar)

$H = (1,2,\ldots d, h_d, h_{d+1},\ldots, h_i \ldots, h_j = t)$,
$0 \leq d - h_d \leq h_d - h_{d+1} \leq \ldots \leq h_{j-1} - h_j \leq t$.
(i.e. $0 \leq -\Delta H_d \leq -\Delta H_{d+1} \leq \cdots \leq -\Delta H_{j+1} = t$).

Ex. $H = (1,2,3,4,5,6,7,6,4,2), d = 7$
$-\Delta H_{\geq d} = (1,2,2,2)$ is nondecreasing.

Parametrization (r=2):
- (Iar) LevAlg(H) irreducible, smooth, dim known.
- Frontier property for closures of strata:

  $\overline{LevAlg(H)} = \bigcup_{H' \leq H}$ LevAlg(H').
- (Chipalkatti, Geramita): structure of special unions of strata.

IC. Why do we care?

- Ubiquity of Gorenstein algebras (Bass, Huneke)

  a. Cohomology rings (often nonstandard grading). (V. Puppe et al): Use "generic" $A, H = 1, 6, 6, 1)$ to create manifold having no S^1 quotient.

  b. Stanley-Reisner rings of complexes that are triangulations of sphere

  c. Commutative algebra: Gor. is strong condition

  d. We care about the higher derivatives of a form F, or polynomial $f = F_j + F_{j-1} + \cdots + F_d$.

  e. Dimension d Gorensteins B are related to their Artinian quotients $A = B/(L_1, \ldots, L_d)$ by a system of parameters L:

    - Minimal resolutions the same
    - H(B) determined by h-seq H(A).

- Level type $t, t \geq 2$, are the next step in taxonomy, "geography" of singularities.

  a. Levels are basic components of graded Artinian algebras:
  $A = R/\cap I(i)$ with $R/I(i)$ level.

b. Dually, write $\hat{A} = \oplus \hat{A(i)}, A(i) = R \circ W(i)$
$W(i) \subset S_{t_i}$.
$W(i) \cong$ dual of degree t_i part of socle.

Build general graded Artin algebras from levels.

c. (T. Hibi): i-skeleton of CM simplicial complex has a level Stanley-Reisner ring.

- Artin algebras occur elsewhere.
  - a. Singularity theory (finite type)
  - b. Punctual Hilbert scheme
  - c. commuting nilpotent sets of matrices (R. Basili, V. Baranovsky, A. Premet, case r=2)
  - d. Algebra: Let (B,m) be any local algebra
    - i. Truncations $A = B/M^n$.
    - ii. $A = B/(L)$ quotient by s.o.p.: minimal reduction of B
  - e. Quivers
    - i. Algebra associated to a quiver
    - ii. quivers with Artin algebras at vertices:
      Ex. Zwara: $k[t]/(t^n)$ “curvillinear”
  - f. Approximation theory/ interpolation (A. Hirschowitz, A. Alexander, K. Chandler, G.G. Lorentz, B. Shekhtman, C. de Boor, G. Ellingsrud, E. Ballico, C. Ciliberto).

Note: “A” *often non-graded* in applications!

Given $H = (1, \dots, h_j, 0)$. let NGLevAlg(H)={ nongraded A | $Soc(A) \to A_j$ is an isom. and $H(A) = H$}

- Except r=2 or t=1, or H "compressed" (maximal), little is known about NGLevAlg(H):
  Q1 NG: What H occur?
  Q2 NG What is dimension of NGLevAlg(H)?

- When $t = 1$, we write NGGor(H). Then $A^* = Gr_m(A)$ has added structure, a sequence of subquotients, $Q(a)$ each with HF symmetric about $(j - a)/2$.

Let $F \in S, F = F_d + F_{d+1} + \cdots + F_j$;
$A_F = R/\text{Ann}(F)$: $Q(a)$ depends on $F_{j-a} + \cdots + F_j$.
$Q(0) = R/\text{Ann } F_j =$ unique soc. deg. j Gor qt of A^*.

Ex. $F = Z^3 + X^4 + Y^4 \in S = k[X, Y, Z]$. Then $Q(0) = R/\text{Ann}(X^4 + Y^4) \cong k[x, y]/(xy, x^4 - y^4)$.
$H(0) = (1, 2, 2, 2, 1)$;
$Q\hat{(}1) \cong \langle Z, Z^2 \rangle$,
$H(1) = (0, 1, 1)$.
$H(A) = H(0) + H(1) = (1, 3, 3, 2, 1)$

Q3 NG. Which $H(Q(a))$ occur when $t = 1$?

•. When $r \geq 3$, despite this symmetry, and the Buchsbaum-Eisenbud Structure Thm for r=3, the answers to Q1NG and Q3 NG. are unknown even for A height three Gorenstein. *This is scandalous!*

ID. “Generic” Level Algebras.

Consider ComAlg(n): commutative algebras A, $\dim_k A = n$, parametrized by structure constants.

Q4 NG. Describe irreducible components.

Ex. (Emsalem-I): H(A)= (1,4,3) local algebra determines a component of ComAlg(8). $R = k[x, y, z, w]$

Choose A=R/I, I=(7 gen. enough quad forms in R). Then A has deformations only to same kind of algebra A’, same H.F. $H(A') = (1, 4, 3)$.

A is not ”rigid” , but is ”generic”: 6 dim. family.

Ex. (Shafarevich): $H(A) = (1, d, e), A$ general.
If $2 < e \leq (d-1)(d-2)/6 + 2 \Rightarrow$ A generic.
If $(d^2 - 1)/3 \leq e \leq \dim R_2 \Rightarrow$ A is “smoothable”.

Def. A smoothable ⇔ A has deformation to $k \oplus \cdots \oplus k$.

- If $d^2/6 \leq e \leq d^2/3$, deformation status open.

Ex. Gorensteins H(A)=(1,r,r,1), $r \geq 6, r \neq 7$ (up to r=13) ⇒ A generic. (Open $r > 13$).

Probable Ex. Many A level compressed (so H(A) max, given (r, t, j)). should be generic.

As $r \geq 5, j \geq 3$, t well chosen.

∃ Small tangent space (STS) argument, but a generalization of the Shafarevich result is needed.

(The “oracle” computer algebra programs have so far refused to answer for an infinite number of cases)

Ex. r=5, H(A)= (1,5,15,15,5,1);
Ex. r=4, A compressed t=1, j=15.

- Deformation and genericity status of level algebras is open in general, except r=2, or r=3, t=1 (for both these cases all Artin A are smoothable).

$*$ Some nonsmoothable A below may be generic points of ComAlg(n). Each deforms to generic B.

Argument below uses Hilbert scheme parameters.

Ex. Let $H = (1, 3, 6, 10, 15, 21, 18, 6)$
Let $A =$ general element of NGLevAlg(H).
Then A is nonsmoothable, by dimension calc:
A compressed level, $j = 7.$, length $n = 80$.
Note $R = k[x, y, z], H(R) = (1, 3, \ldots, 28, 36, \ldots)$.
dim NGLev(H)$= 6(28 - 18 + 36 - 6) = 240$, as
$F(1) = F(1)_6 + F(1)_7, \ldots., F(6) = F(6)_6 + F(6)_7$.
Let $U(3, n) =$ smooth subschemes of $\mathbf{A}^3$.
Dim $U(3, 80) = 3(80) = 240$.
$\Rightarrow A$ has a "generic" non-smooth deformation B.

Ex. Let $H = (1, 5, 15, e), 4 \leq e \leq 26$.
$\exists A \in$ LevAlg(H) that are non-smoothable.

If $6 \leq e \leq 18$: $\exists A \in$ LevAlg(H), candidates to be "generic" points of components of ComAlg(n) via small tangent space (STS) argument.

Ex. $H = (1, 6, 21, e)$ $4 \leq e \leq 46$, $\exists$ A nonsmooth.

If $6 \leq e \leq 32$: $\exists A$ candidates to be "generic" points of ComAlg(n) via STS argument.

II. Which Hilbert functions occur?.

IIA. Recent progress for level algebras, $t > 1$.

- A. Geramita, J. Migliore et al: classified $H, r = 3, t = 2, j \leq 6$. Many techniques (AMS Memoir)
- A. Geramita & A. Bigatti, Cho & I.:
  determined extremal H.

Q1. Can $H(A)$ be non-unimodal, r=3, $t > 1$?

Q1 Ans. (F. Zanello $t = 28$, then A. Weiss $t = 5$) Yes!

- F. Zanello:
  - (a) $\exists$ non-unimodal $H, r = 3, t = 28$ (see below).
  - (b) Also, $\exists H$ with arbitrarily many dips.
- A. Weiss: $\exists$ non-unimodal $H, (r, t, j) = (3, 5, 63)$; also $(r, t, j) = (4, 4, 17)$.

Q1' Are there smooth pts Z in $\mathbf{P}^3$ with non-unimodal h-vector?

Q1' Ans (J. Migliore) Yes! (*Open for many dips*)

IIB. Zanello's non-unimodal $H, r = 3$.

Def. Let $H(1), H(2)$ be Hilbert functions of level algebras. Let $r_i = \dim_k R_i$. Then

$$(H(1) +_h H(2))_i = \min\{H(1)_i + H(2)_i, r_i\}, r_i = \dim_k R_i.$$

Lemma 2. (Iar.) Let $W \subset S_j$, and choose F generic in S_j, set $W' = \langle W, F\rangle$. Then

$$(H_F)_i = \min \{r_i, r_{j-i}\},$$

$$H_{W'} = H_W +_h H_F.$$

Here $R/Ann(F)$ is compressed Gorenstein,

$R/Ann(W')$ relatively compressed with respect to $R/AnnW$ (max possible $H_{W'}$).

Thm. (Zanello, 2005) $\exists$ Non-unimodal level Gor. seq. $H_{NU} = (1, 3, 6, 10, 15, 21, 28, 27, 27, 28)$.

Proof: First, a truncated Gorenstein sequence is a level sequence (by Lemma 1).

a. Consider Gor. sequence $H_G, G \in S_{18}$,

$H_G = (1, 3, 6, 9, 12, 15, 18, 21, 24, 27, 24, \ldots.., 1)$,

Truncate at 27 (j=9), to get $H(1)$.

Let $W = R_9 \circ G$. Then $H_W = H(1)$.

b. Take $F \in S_9$ generic,

Then $H_F = (1, 3, 6, 10, 15, 15, 10, 6, 3, 1)$.

c. Set $W' = \langle W, F\rangle$. Apply Lemma 2.

$$\begin{aligned} \text{Then } H_{W'} = {} & (1, 3, 6, 9,\ 12, 15, 18, 21, 24, 27) +_h \\ & (1, 3, 6, 10, 15, 15, 10,\ 6,\ 3,\ 1) = \\ & (1, 3, 6, 10, 15, 21, 28, 27, 27, 28) = H_{NU}. \end{aligned}$$

Ex. (Weiss-I.) r=3, t=5, j=63. Take four general forms G_1, G_2, G_3, G_4 in $[(x^{21})^{\perp}]_{63}$, and one generic form F, and, similarly, use Lemma 2.

Weiss shows: $R \circ \langle G_1, ..., G_4\rangle$ and $R \circ W'$ have the expected HF's. This generalizes the compressed algebra construction.

Weiss gives a general method of constructing similar non-unimodal examples, but cannot with this method obtain smaller t, given r.

Best via Weiss-I.: $r = 3, t = 5; r = 4, t = 4$. Recall $r = 3, t = 1$ unimodal; $r = 4, t = 1$ unimodality is open.

Note. When $r \geq 5, \exists$ non-unimodal Gor seq. H.

Ex (Stanley), $r = 13, H = (1, 13, 12, 13, 1)$.
Let $R = k[u_1, \ldots, u_{10}, x, y, z]$,
$S = k[U_1, \ldots, U_{10}, X, Y, Z]$
$F = \sum_{i=1}^{10} U_i\mu_i = U_1X^3 + \cdots + U_{10}Z^3$
$\mu = (X^3, X^2Y, X^2Z, XY^2, \ldots Z^3)$.
$R_2 \circ F \subset (x^2, xy, \ldots, z^2) \circ F + \langle X^2, XY \ldots Z^2 \rangle$.
(multiple of u_i acting).
So $H_2 \leq 12$.
(Reiten construction: $A = B \times \hat{B}$, $B = k[x, y, z]/M^4$.)

Ex (Bernstein-I) $r = 5, j = 16$
$H = (1, 5, 12, 22, 35, 51, 70, 91, 90, 91, \ldots, 1)$.
Let $S = k[U, V, X, Y, Z]$,
$F = UF_1 + VF_2$,
$F_i \in k[X, Y, Z]_{15}$, general.
Reiten construction: $A = B \times B$,
$B =$ compressed algebra in $\mathrm{LevAlg}_{k[x,y,z]}(2, 15))$.

$H(B) = (1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 42, 30, 20, 12, 6, 2)$.
$H(\hat{B}) = (-, 2, 6, 12, 20, 30, 42, 55, 45, 36, \quad \ldots \quad, 10, 6, 3, 1)$.
$H(A) = (1, 5, 12, \quad \ldots, \quad 91, 90, 91, \quad \ldots \quad, 12, 5, 1)$.

$\exists$ Similar examples $(r, j) = (6, 10), (7, 8)$;
$(r, 6), 8 \leq r \leq 21$. Further study by M. Boij-D. Laksov.

Ex. (M. Boij): $\exists$ Gor. sequences with arbitrarily many dips/maxima. Cost: large r.

Q2. What is lowest t for $r = 3, r = 4$, for which $\exists$ level algebra A, type(A) $= t$, $H(A)$ non-unimodal?

Regard as first step to understand HF's.

Q2'. For $r = 3, 2 \leq t \leq 4$ are there non-unimodal level HF's?

For $r = 4, 2 \leq t \leq 3$, are there non-unimodal level HF's?

Q3: For $r = 4$ characterize Gorenstein sequences. Are they unimodal?

Thm. (Srinivasan-I.): Yes: $H = (1, 4, s, \ldots), s \leq 7$.

Proof uses minimal resol, Gotzmann-Grothendieck Hilbert scheme thm.

- Work in progress: (J. Migliore, U. Nagel, F. Zanello.)
  They believe answer is "yes".

Of course, there are many finer questions about HF:

Q4. How does the existence of non-unimodal level HF depend on (r, t, j)?

Q5. Characterize level HF, $r = 3, t = 2$, or small.

Tools from A. Geramita, T. Harima, J. Migliore, Y. S. Shin.

F. Zanello , (I.- case $t = 2$) connects Hilbert function of generic level quotients of type s of A with $H(A)$ of type t.

III. Parametrization: LevAlg(H).

IIIA. When $r = 4, t = 1$, Gor(H) has several irreducible components in general. Here $R = k[x, y, z, w]$.

Ex. (Boij) Let $H = (1, 4, 7, 10, 13, 13, 13, 13, 10, 7, 4, 1)$. Then Gor($H$) has several irreducible components.
C_1: $A = R/I_Z, Z$ punctual schemes, 13 points on a rational normal curve, where $H_Z = (1, 4, 7, 10, \underline{13})$.
C_2: Ideals $I = Ann(F), F \in S_{11} \mid I \supset I_W^2$, where I_W defines a line W in $\mathbf{P}^3$.

Ex. (J.O. Kleppe) Let $H = (1, 4, 9, 14, 19, 19, 14, 9, 4, 1)$. Then Gor($H$) has at least three irred. comp., algebras with same Betti nos. (minimum in $\beta(H)$). This example lifts to points Z in $\mathbf{P}^4$, $h(Z) = H$.

Ex. (I.-Srinivasan) Let $H = (1, 4, 7, h, b, \ldots)$,
$8 \leq h \leq 10$, $3h - b - 17 \geq 0 \Rightarrow$ Gor(H) has several components. Proof uses new structure theorem when $I_2 = (wx, wy, wz)$.
As $H = (1, 4, 7, 9, 10, 9, 7, 4, 1)$.

- $r \geq 4$ Study of weak Lefschetz level sets of points (J. Migliore, R. Miro-Roig).

IIIB. OUR PROBLEM (work with M. Boij)
- When $r = 3$: Do $\exists$ LevAlg(H), t small, reducible?
- If so, are they liftable to families of points?

(Recall $r = 3, t = 1$, Gor(H) irreducible by S. Diesel, so reducibility of LevAlg(H) for $r = 3, t \geq 2$ was open.)

i. Example. $H = (1,3,4,4)$. Then LevAlg(H) has two irreducible components, each of dimension 8.

$R = k[x,y,z], H(R) = (1,3,6,10,\ldots)$.

C_1: CI-related:
$I = (f,g,M^4), f,g \in R_2$ and $R/(f,g)\, CI$.
Dimension: $2(4) = 8$.
Example: $I = (x^2,y^2,M^4)$.

C_2: I_2 has common factor.
$I = (xy,xz,f,M^4), f \in R_3$.
$H(R/(xy,xz)) = (1,3,4,5,\ldots)$.
Dimension $C_2(H) = 2+2+4 = 8$, for choice of $x \in R_1, \langle y,z\rangle \subset R_1$, and $f \in R_3/R_1I_2$.
Example: $I = (x^2,xy,z^3,M^4)$.

Lemma: C_1, C_2 are irreducible components of LevAlg(H), H=(1,3,4,4).
Also $C_1 \cap C_2 = \{(xV, hv, M^4)$
s.t. $v \in V, V \subset R_1$, dim $V = 2, h \in R_2\}$.

Example; Since $h \in R_2/(xR_1)$, dim $C_1 \cap C_2 = 7$. $I = (x^2,xy,y^3,z^4,z^3y,z^3x) \in C_1 \cap C_2$.

Proof A. C_1, C_2 have equal dimension, and cannot be in the closure of another component by (B.)

Proof B. $\exists$ two distinct minimal elements $\beta(C_1), \beta(C_2)$ in POS $\beta_{level}(H)$, partially ordered set of minimal graded level Betti sequences compatible with H. (See [Ragusa-Zapallá],[Peeva] and [Migliore] for this argument).

Betti sequence for C_1:

```
total   1  6   9   4
  0:    1  -   -  -
  1:    -  2   -   -
  2:    -  -   1  -
  3:    -   4   8  4
```

Betti sequence for C_2:

```
total   1  6   9   4        Note:
  0:    1  -   -  -          0:     1   -    -  -
  1:    -  2   1   -          1:     -   2    -   -
  2:    -   1   -  -         2:     -   -    -  -
  3:    -   3   8  4          3:      ...
                                is not possible.
```

$\exists$ no sequence smaller than both in POS $\beta_{level}(H)$:
as $\langle f, g\rangle$ has no linear relation $\Rightarrow \exists$ quadratic relation.

Q6. Can one lift Artinian examples of H such that LevAlg(H) has several irreducible components to similar examples for the postulation Hilbert scheme of (smooth) points having h-vector H?

Answer. Yes, here and in several other cases.

Q7. (Stanley, Hibi) Is there a graded Gorenstein domain with with non-unimodal h-vector? (*Open*)

Q7'. Is there a smooth punctual arithmetically Gorenstein scheme Z with non-unimodal h-vector? (*Open*).

Notation: Z a zero-dimensional scheme in P^3,
$T = H_Z$ postulation, $H_Z = \Delta T = h$-vector of Z.
Z is *level* if min. reduction $R/(\ell, I_Z)$ is level Artin.
LevPoints(H): level Z with $H_Z = H$.

Theorem. Let $T = (1,4,8,12,12,\ldots), H = (1,3,4,4)$. LevPoints(H)$\subset Hilb^{12}P^3$ has two irreducible components C_1, C_2, each of dimension 28. Here $R = k[x,y,z,w]$.

C_1: 12 points on the CI intersection of two quadrics:
dim $C_1 = (2 \cdot 8) + 12 = 28$.

C_2: 12 points: a cubic surface S intersection a line L, union nine points lying on a hyperplane section $S \cap P$ of S. Here we take $S \cap L = (y,z,f), S \cap P = (x,f)$:

$$I = (xy, xz, f, g_1, g_2, g_3),\ f \in R_3,\ g_u \in R_4.$$

$$H(R/(xy,xz)) = (1,4,8,13,\ldots).$$

dim C_2 = choose x, then (y,z), then f, then 9 pts.
dim C_2 = dim P^3+dim Grass$(2,4) + 12 + 9$
$= 28$.

Proof: C_1, C_2 nonempty (a level smooth Z is possible) by lifting of monomial ideals in the corresponding components of LevAlg(H).

Since the Betti numbers of C_1, C_2 are each minimal in $\beta_{level}(H_Z)$, there is no bigger component specializing to C_1, C_2.

ii. Type three family of Hilbert functions $H(c)$:

$H(3) = (1, 3, 6, 8, 9, 3)$,
$H(4) = (1, 3, 6, 10, 13, 15, 9, 3), \ldots..$
$H(c) =$ maximum Hilb function for type 3, socle degree $j = 2c - 1$ quotient of $\mathrm{CI}(c, c)$, $R = k[x, y, z]$:

$$H(c)_i = r_i \text{ for } i < c,\ H(c)_c = r_c - 2.$$
$$H(c)_i = max\{r_i - 2r_{i-c}, 3r_{2c-1-i}\} \text{ for } i > c.$$

The component $C_1(c)$ of $H(c)$ parametrizes type 3, socle degree j=2c-1, quotients A=B/J, of CI algebras $B = R/I$,

$$I = (f, g),\ \deg f = \deg g = c,$$

The component $C_2(c)$ parametrizes type 3, socle degree j quotients of algebras B=R/I

$$I \cong (hx, hy, f), h \in R_{c-1}, f \in R_{c+1}.$$

Theorem. (with M. Boij)
$H(c)$ has two irreducible components $C_1(c), C_2(c)$, each of dimension $4c^2 + 3c - 11$.

Proof. A. Show HF $H(c)$ is attained by type three level quotients for $C_1(c), C_2(c)$. For C_1 use a lemma about type t quotients of sets Z of points, having UPP (uniform position property), when $j \geq \sigma(Z)$ regularity. For C_2 study quotients of almost UPP points.
B. Easy dimension calculation. (See below, case $c = 3$).
C. We show $\exists$ no common refinement (smaller) Betti numbers possible in $\beta_{level}(H(c))$: for A general, $A \in C_1(c)$, we have $\beta(A) = \beta_{min}(H(c))$.

Note. These appear to lift to points (*in progress*).

Ex. $H = (1,3,6,8,9,3)$. First $c = 3$ in $H(c)$ series above. Dimenson calculation of components.

a. Open $U_1 \subset C_1$ component. Choose $(f,g) \in \mathrm{CI}(c,c)$.
$H(R/(f,g)) = (1,3,6,8,9,9,\ldots)$.

Then choose 6-diml subspace $V \subset B_5, B = R/(f,g)$
(or choose 3-diml subspace $W \subset {(f,g)_5}^{\perp} \subset S_5$).

Then $A = R/\mathrm{Ann}\ W = B/V \in U_1$.
So V is an element of Grass$(6,9)$, and
dim $U_1 = 2 \cdot 8 + 6 \cdot 3 = 34$.

b. Open $U_2 \subset C_2$ component.

Choose $h \in R_2, f \in (R/(xh,yh))_3$.
$B = R/(xh,yh,f), H(B) = (1,3,6,8,9,9,\ldots)$.

Then choose 6-diml subspace $V \subset B_5$,
(or choose 3-diml space $W \subset {(xh,yh,f)_5}^{\perp}$).

Then $A = R/\mathrm{Ann}\ W = B/V \in U_2$.

choose h, (x,y), f, V
dim $U_2 = 5 + \ 2 + \ 9 + \ 6 \cdot 3 = 34$.

Betti for $A \in U_2$:

```
 total   1   9   11   3
   0:    1   -    -  -
   1:    -   -   -   -
   2:    -   2    1    -
   3:    -   1    -  -
   4:    -   6   10   -
   5:    -   -    -    3
```

For $A \in U_1$ omit the diag $(1,1)$.

iii. Ex. (Boij-I, 5/2006) $H = (1,3,6,10,12,12,6,2)$: LevAlg(H) has two irred comp.

- C_1: type two quotients of coord ring of B of 12 general pts. on P^2. Dim $C_1 = 24 + 20 = 44$.

- C_2 type two quotients of coord ring B' of 13 pts cutting out only 12 conditions on quartics.

  Dim C_2=23+2(11)=45.

Proof. The Betti sequences for general pts of C_1, C_2 are minimal elements of $\beta_{level}(H)$.
Uses min resolution, Gotzmann-Grothendieck Thm.

Betti sequence for C_1:

```
total   1   9   10   2
  3:    -   3    -   -
  4:    -   -    2   -
  5:    -   6    8   -
  6:    -   -    -   -
  7:    -   -    -   2
```

Betti sequence for C_2:

```
total   1   8   9   2
  3:    -   3   1   -
  4:    -   1   -   -
  5:    -   4   8   -
  6:    -   -   -   -
  7:    -   -   -   2
```

Appears to be part of an infinite irreg. family.
Lifting to points undone.

iv. Type two family of Hilbert functions $H(s,k)$.
$H(1,0) = (1,3,6,10,15,21,27,30,30,20,12,6,2)$.
$H(0,1) = (1,3,6,10,14,18,20,20,12,6,2)$.

Assume $(s,k) > (0,0)$.

Let $\nu = 2s+4, j = 5s+3k+7, r_i = \dim R_i$.

$H(s,k)$: the Hilbert function of type two, socle degree j quotients of the coordinate ring of $2r_{\nu-2+k}$ points lying on a degree ν curve in P^2.

$C_1(s,k)$: as in definition of $H(s,k)$.

$C_2(s,k)$: Consider schemes X comprised of a general degree $\nu/2 = s+2$ curve Y in P^2 union a punctual scheme Z comprised of $n = 3r_s+\nu\cdot k$ points of P^2 having h-vector

$$S = (1,2,\ldots,s+1,s+2,\ldots,s+2,0),$$

with exactly $s+2k+1$ occurences of s+2. Then $C_2(s,k)$ is family of type two socle degree j
quotients of R/I_X's.

Conjecture. (with M. Boij) $C_1(s,k), C_2(s,k)$ are two irreducible components of LevAlg(H(s,k)).

$\dim C_1(s,k) = 28r_s + 7 + 11(s+1) + 6\nu\cdot k$.
$\dim C_2(s,k) = \dim C_1(s,k) + r_s - 1$.

(*A,B in plan of proof are clear, but C* needs work!*).

Proof outline:
A. Show that the expected Hilbert functions are achieved in each construction.

B. Show the dimension formulas, using Gotzmann result: dimension of postulation strata of $\mathrm{Hilb}^n P^2$.

C*. Show that there is no common refinement in Betti(H(s,k)) of the Betti sequences giving these two constructions.

Or

Show that the tangent space Hom(I,A) to $A \in C_1(s,k)$ is the same as the dimension of $C_1(s,k)$. Use construction by J.O Kleppe, M. Boij-I. of
tangent space to level algebras, especially simple in type 2.

Note. We are working on liftability to points, likely using a very recent result of Migliore.

v. Expected Example. Type two, outside of the above series:

$$H = (1,3,6,10,15,21,27,33,39,45,50,54,54,\\ 42,30,20,12,6,2)$$

$C_1(H)$= level quotients of 54 points lying on a degree six curve in P^2, with h-vector

$$S = (1,2,3,4,5,6,6,6,6,5,4).$$

$C_2(H)$= level quotients of scheme X = degree four curve union 12 points on P^2 having h-vector $S = (1,2,2,2,2,2,1)$.

Here dim $C_1(H) = 180$, dim $C_2(H) = 199$ but cannot specialize from $C_2(H)$ to $C_1(H)$.

Conclude: There are many examples of type two level Hilbert functions $H = (1,3,\ldots,2)$ for which there are at least two irreducible components of LevAlg(H). These can be systematically constructed using the Gotzmann dimension result by showing that a "more special" component $C_2(H)$ has dimension greater or equal that of $C_1(H)$.
The proof of each example also requires a smoothness-of-$C_1(H)$, or Betti argument to verify.

This situation for level type 2, $r = 3$ is in stark contrast to level type 1 where Gor(H) is irreducible.

IV. Overview, open problems.

IVA. Overview, what do we want to learn?

i. Gorenstein Artinians: widely studied, relatively understood for $r = 3$. Much unknown for $r > 3$, including possible Hilbert functions
(see Q1 NG,Q3 NG for r=3, Q3,Q7 above),
Betti strata $r = 3$: Boij. Closures of $\mathrm{Gor}(H)$, $r = 3$ partially determined by Kanev-I.

Open in general.

ii. When r=2, $\mathrm{LevAlg}(H)$ well understood for $r = 2$; HF: I-, Froberg, Chipalkatti-Geramita).
Frontier property for closures (I-). Connection to secant varieties of RNC (see [IK], SLN # 1721).

We don't expect such good results for $r > 2$*, but what can be said?*

iii. When r=3, t=2. First case beyond $r = 2$. That $\mathrm{LevAlg}(H)$ may have several irred components was resolved above (Boij-I.) and below (J.O.Kleppe).

Q8 (Migliore): Let (r, t, j) =(# vars, type, socle degree), set

$n(r, t, j) = \max_H\{\#$ components of $\mathrm{LevAlg}(H)$,
$H \in \mathrm{LevAlg}(r, t, j)\}$.

Find dependence of $n(3, 2, j)$ on j. In particular, does $n(3, 2, j) \to \infty$ with j?

Q9. Can we use Gorenstein quotients of A to show there exist several components in $\mathrm{LevAlg}(H)$?
By Macaulay duality:

LevAlg$(H)\subset$ Grass $(t, S_j), S = k[X_1, \dots, X_r]$.

When $t = 2$, A type 2 $\Leftrightarrow V \subset S_j$ dim V=2. Consider the pencil of forms $f \in V$. Then

$\{A_f = R/\text{Ann } f, f \in V\}$

is a pencil of GA quotients of A. Look at unusual pairs $(f, g), \langle f, g\rangle = V$, where $H(A_f)$, $H(A_g)$ takes extremal values, as an invariant of A.

We expected that different constructions of $A \in$ LevAlg(H), using different Gorensteins could lead to several components, and we conjectured:

Ex. (J.O. Kleppe) $H = (1, 3, 6, 10, 14, 10, 6, 2)$.
LevAlg(H) has several irreducible components. Proof uses behavior of deformation spaces under linkage, (technique pioneered by Buchweitz).

Is there a proof using the decompositions of H? As,

$$\begin{array}{ll} (1,3,5,7,7,5,3,1) & (1,3,4,5,5,4,3,1) \\ +(1,3,5,7,7,5,3,1) & +\,(1,3,6,9,9,6,3,1) \\ (1,3,6,10,14,10,6,2) & (1,3,6,10,14,10,6,2) \end{array}$$

Note: $\exists$ Natural inequalities among $H(A), H(A_f), H(A_g)$ for $A = Ann(f, g)$.
(I. generalized by F. Zanello to level type s quotients of level type t algebras).

Can these be used to restrict the sequences H possible as $H(A)$ for level algebras?

Q10. What makes a good study of LevAlg(H)?

Irreducible components, dimension, tangent space, liftability of components to LevPoints(T), $\Delta T = H$. Closures and intersections of components.

IVB. Deformations. (See also §1C).

Establishing the tie to schemes in $\mathbf{P}^{r-1}$ seems the most salient feature of recent work by M. Boij,
I.-V. Kanev, J.-O. Kleppe, J. Migliore & R. Miró-Roig, I.-Srinivasan.

Let $\sigma(B) =$ regularity degree of B.

Thm (J.O.Kleppe) Let $U =$ family of cod c, dim one CM quotients B of R having Hilbert function H_B, that satisfy certain conditions. Let $j \geq 2\sigma(B)$, and let $H_A = Sym(H_A, j/2)$. Consider the family W of weak-Lefschetz Gorenstein algebras of Hilbert function H_A. Then there is an inclusion of the set of components of U to those of W, such that generically smooth components of U map to generically smooth components of W.

Summary: $\exists$ good map from postulation Hilbert scheme (fix h-vector H_B), to Gor(H_A).

Q11. Are there rigid non-smooth Artinian algebras A (so A is not $k \oplus \cdots \oplus k$, and has deformation only to A)?

IVC. Mostly accessible problems:

(See also list in I-Kanev SLN 1721 Ch. 9).

A. What are Gor. Seq. $H, r = 4, t = 1$?

B. Cohomology ring of proj. manifolds G_H =GrAlg(H), $r = 2$. Same for $\overline{LevAlg(H)}, r = 2$. The additive cohomology groups of G_H are known, and are related to a coding of partitions of diagonal lengths H (I.-Yameogo).

C. Closure of postulation strata $Z(H) \subset Hilb^H(P^2)$

D. Closure of $PGor(H), r = 3$: (see SLN 1721).

E. Classification: $r = 3, t \geq 2$: What H occur for level algebras? What are components of LevAlg(H)?

F. Tighten connection with singularities. The classification of singularities of finite type by specialists has apparently not reached (co)lengths n where interesting behavior involving components of CommAlg(n) is common, nor has the structure of NGGor(H) been used.

G. Nongraded level algebras, Gorenstein algebras: what are H, r=3? (see Q1 NG, Q3 NG).

H. Quantum topologies and Gorenstein algebras

I. Hilbert scheme of points:

A. on $\mathbf{P}^2$ or surfaces (large recent development, connections to Heisenberg algebras, combinatorics). Deep results of Haiman et al.

B. on $\mathbf{P}^r$: component structure? Connections to component structure for commuting nilpotent matrices. (see Huibregtse, Guralnick).

J. Hilbert schemes and ideals of extremal growth.

(see I.- S. Kleiman Appendix C to SLN 1721).

K. Closure of strata $Z(H)$ (all ideals), in $Hilb^n k\{x,y\}$.

L. Study of $\mathrm{Hilb}^n(k[x,y]/J)$: ideals I of colength n in $R = k[x,y]$ or $k\{x,y\}$ containing a given ideal J.

Thm (Shoshistvili): The local punctual Hilbert scheme $\mathrm{Hilb}^n(k\{x,y\}/J)$ is $\mathbf{P}^1$-connected.

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