An algebra of commuting nilpotent matrices

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Abstract:

Let $Mat_n(K)$ denote the ring of $n \times n$ matrices over a field K. Fix a nilpotent $n \times n$ matrix B of Jordan partition P, and consider the centralizer $\mathcal{C}_B$ of B, and its subvariety $\mathcal{N}_B$ of nilpotent matrices. Denote by $N^2(n, K)$ the variety of commuting pairs of nilpotent matrices. We describe recent work on both these varieties, and the connections with previous work by J. Briançon et al on the fibre $H^{[n]}$ of the punctual Hilbert scheme $Hilb^n(P^2)$ of the plane over a point $p \in P^2$.

R. Basili defined a maximal nilpotent subalgebra $\mathcal{U} = \mathcal{U}_B$ of $\mathcal{N}_B$. We describe an involution on $\mathcal{C}_B$, and give bases for the quotients $\mathcal{U}^i/\mathcal{U}^{i+1}$.

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Note. The natural connection between commuting $n \times n$ nilpotent matrices and the fibre of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ over a point p of $\mathbb{A}^2$ was noted by H. Nakajima; and used by V. Baranovsky, R. Basili, and A. Premet, to study the irreducibility of the variety of pairs of commuting nilpotent matrices [Bar2001], using J. Briançon's work, or vice versa. Since a pair of commuting matrices may not have a cyclic vector, the theory of pairs and triples of commuting nilpotent matrices is related to that of Hilbert schemes, but is not "isomorphic".

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Preface: Consider a subalgebra $\mathcal{A} = \mathcal{A}_{S,T} \subset \mathrm{Mat}_n(K), K$ a field, comprised of nilpotent matrices A, and defined by a set S (zeroes of entries), and by equalities T among entries.

i. $\exists\, S \subset [1,n] \times [1,n] \mid A_{ij} = 0$, for $(i,j) \in S$.

ii. $\exists\, T = (T_1, \ldots, T_t), T_i \subset S^c = [1,n] \times [1,n] - S$ such that for each $i, 1 \leq i \leq t, \quad A_{uv} = A_{u'v'}$ when $(u,v), (u,v') \in T_i$.

Lem 0.1. *$\mathcal{A}$ and each $\mathcal{A}^k, k \in \mathbb{N}$ are irreducible sets.*

Question 1. What is the rank of the general element of $\mathcal{A}^k$?

Question 2. Let A be a generic in $\mathcal{A}$. What is rank A^k?

Question 2'. What is the partition P_A determined by the Jordan blocks of A for a generic $A \in \mathcal{A}$?

When $T = \emptyset$, these were answered differently by R. Gansner, and S. Poljak in terms of the digraph $\mathcal{D}(\mathcal{A}) =_{def} \mathcal{D}(A)$ for a generic $A \in \mathcal{A}$ (i.e. $\mathcal{D}(S^c)$).

Definition 0.2. *Digraph* $\mathcal{D}(A)$ of a matrix $A \in M_n(K)$: Directed graph:

Vertices $= \{1, 2, \dots n\}$; An arrow from i to j iff $A_{ij} \neq 0$.

Def. Two k-walks $W = (w_1, \dots, w_k)$, $W' = (w'_1 \dots, w'_k)$ on $\mathcal{D}$ are *vertex independent* if for each $i, 1 \leq i \leq k$, $w_i \neq w'_i$.

Thm 0.3. *[Ga, Pol] Assume* $T = \emptyset$*.*

i. (Gansner) Consider the sequences

$C = (c_1, c_2, \dots), c_i =$ *max # distinct vertices covered by* i *chains of* $\mathcal{D}$*;*

$D = (d_1, \dots,), d_i =$ max # vertices covered by

i antichains.

Then for A generic in $\mathcal{A}, P(A) = \Delta C, P^\vee(A) = \Delta D$.

ii. (Poljak) The maximum rank of $A^k, A \in \mathcal{A}$, and also of $A' \in \mathcal{A}^k$ is the maximum number of (vertex) independent k-walks in the digraph $\mathcal{D}(\mathcal{A})$.

For (i.) see also T. Britz and S. Fomin [BrFo].

For (ii.) see also H. Knight and A. Zelevinsky [KnZe]

Open: Find rank A^k concisely for specific algebras $\mathcal{A}$.

Problem: Generalize Poljak and Gansner's Thm. to $T \neq \emptyset$.

Ex 0.4. Consider the algebra $\mathcal{A}_S \subset \mathrm{Mat}_5(K)$: here the zero entries are S, starred $\{*\}$ entries form S^c, and $T = \emptyset$.

$$\mathcal{A}_S: \quad A = \left(\begin{array}{ccc|cc} 0 & * & * & * & * \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & 0 & * \\ 0 & 0 & * & 0 & 0 \end{array}\right)$$

[The next page is handwritten digraph for this $\mathcal{A}$, after P. Oblak [Ob2]]

1 What is $Q(P)$, maximum nilpotent orbit in $\mathcal{N}_B$?

Let $K =$ algebraically closed field, $\mathrm{Mat}_n(K) = n \times n$ matrices.

$\mathcal{N}_n(K) = \{$nilpotent $A \in \mathrm{Mat}_n(K)\}$.

Fix $B \in \mathcal{N}_n(K)$ Jordan, of partition $P_B = (\lambda_1 \geq \cdots \geq \lambda_t)$.

$\mathcal{C}_B = \mathcal{A} \in \mathrm{Mat}_n(K) \mid [A, B] = 0.$ $\quad \mathcal{N}_B = \mathcal{C}_B \cap \mathcal{N}_n(K)$.

Problem 1.1. Find $\mathcal{Q}(P) = \{$Jordan partitions of $A \in \mathcal{N}_B\}$.

Ex 1.2. $P = (4)$, so B is *regular* (single Jordan block).

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

When $a \neq 0$, $A^3 \neq 0$ and $P_A = (4)$.

When $a = 0, b \neq 0, A^2 = 0$, $\mathrm{rk} A = 2$, $P_A = (2, 2)$

When $a = b = 0, c \neq 0$, then $P_A = (2, 1, 1)$.

When $a = b = c = 0$ then $P_A = (1, 1, 1, 1)$.

$(3, 1) \notin \mathcal{Q}(P)$ *for* $P = (4)$.

1.1 The morphism $\pi : \mathcal{C}_B \to \mathcal{C}'_B$. ($\mathcal{C}'_B$ semisimple)

R. Basili [Bas1] using [TurAi] parametrized $\mathcal{N}_B$, and $\mathcal{U}_B$

Ex 1.3. Let $P = (3, 3, 2), B = J_P$. Then $A \in \mathcal{C}_B$ satisfies:

$$A = \left(\begin{array}{ccc|ccc|cc}
\underline{a_{11}^1} & a_{11}^2 & a_{11}^3 & \underline{a_{12}^1} & a_{12}^2 & a_{12}^3 & a_{13}^1 & a_{13}^2 \\
0 & a_{11}^1 & a_{11}^2 & 0 & a_{12}^1 & a_{12}^2 & 0 & a_{13}^1 \\
0 & 0 & a_{11}^1 & 0 & 0 & a_{12}^1 & 0 & 0 \\
\hline
\underline{a_{21}^1} & a_{21}^2 & a_{21}^3 & \underline{a_{22}^1} & a_{22}^2 & a_{22}^3 & a_{23}^1 & a_{23}^2 \\
0 & a_{21}^1 & a_{21}^2 & 0 & a_{22}^1 & a_{22}^2 & 0 & a_{23}^1 \\
0 & 0 & a_{21}^1 & 0 & 0 & a_{22}^1 & 0 & 0 \\
\hline
0 & a_{31}^2 & a_{31}^3 & 0 & a_{32}^2 & a_{32}^3 & \underline{\alpha_{33}^1} & a_{33}^2 \\
0 & 0 & a_{31}^2 & 0 & 0 & a_{32}^2 & 0 & \alpha_{33}^1
\end{array}\right)$$

with entries in the ring $\mathbb{Z}[a_{11}^1, \ldots, a_{33}^2]$ in 22 variables. Let

$\mathfrak{J}$ = Jacobson rad. of $\mathcal{C}_B$, $\mathcal{C}'_B = \mathcal{C}_B/\mathfrak{J}$ semisimple quotient.

Set $\mathcal{A}(3) = \begin{pmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{pmatrix}, \quad \mathcal{A}(2) = (\alpha_{33}^1),$

Morphism: $\pi : \mathcal{C}_B \to \mathcal{C}'_B : A \to (\mathcal{A}(3), \mathcal{A}(2))$.

Note: $\mathcal{N}_B = \pi^{-1}(\mathcal{N}_2(K), 0)$. $\mathfrak{J} = \pi^{-1}(0, 0)$.

Let $\mathcal{U}_B = \pi^{-1}\left(\begin{pmatrix} 0 & a^1_{12} \\ 0 & 0 \end{pmatrix}, 0\right)$, nilp. subalgebra of $\mathcal{N}_B$.

Let $P = (p_1^{r_1}, \dots, p_s^{r_s}), p_1 > \cdots > p_s$.

$\mathrm{Mat}_{\vec{r}} = \mathrm{Mat}_{r_1}(K) \times \cdots \times \mathrm{Mat}_{r_s}(K)$,

$\mathcal{N}_{\vec{r}} = \mathcal{N}_{r_1}(K) \times \cdots \times \mathcal{N}_{r_s}(K)$. $\pi : \mathcal{C}_B \to \mathcal{C}'_B$.

Lem 1.4. *[Bas1]:* $\mathcal{N}_B = \pi^{-1}(\mathcal{N}_{\vec{r}})$ *and is irreducible.*

Def. Denote by $Q(P)$ the partition of a generic $A \in \mathcal{N}_B$.

1.2 Maximal nilpotent subalgebra $\mathcal{U}_B$ of $\mathcal{C}_B$.

Let $\mathcal{U}_{\vec{r}} = \mathcal{U}_{r_1}(K) \times \cdots \times \mathcal{U}_{r_s}(K)$ (s.u.t.). Let $\mathcal{U}_B = \pi^{-1}(\mathcal{U}_{\overline{r}})$.

Lem 1.5. $\mathcal{U}_B$ *is a maximal nilpotent subalgebra of* $\mathcal{C}_B$. *Each element of* $\mathcal{N}_B$ *is similar to an element of* $\mathcal{U}_B$ *under the conjugation action of* ${\mathcal{C}_B}^*$.

Cor 1.6. $Q(P)$ *is the partition of a generic element of* $\mathcal{U}_B$.

Warning. There is no simple analogue of Lemma 1.5 for pairs $A, A' \in \mathcal{N}_B$.

We denote by $\mathcal{D}(P)$ the digraph of a generic A in $\mathcal{U}_B$.

Lem 1.7. *$\mathcal{D}(P)$ has no loops. If $A \in \mathcal{U}_B$ is generic then $\forall k \in \mathbf{N}, \forall i, j \mid 1 \leq i, j \leq n,\ (A^k)_{ij} = 0 \Rightarrow (A^{k+1})_{ij} = 0$.*

Question 3. Is the rank of $A^k, k = 1, 2, \ldots$ an invariant of $\mathcal{D}(P)$? Is this rank the same as that for a generic matrix of zeros and variables with the same digraph [Pol, KnZe]?

(Can we ignore the equalities among entries in finding $Q(P)$?).

1.3 What we know about $Q(P)$ – brief look.

Def. A *string* or *almost rectangular* subpartition of P is one s.t. largest - smallest part ≤ 1.

Let $r_P =$ minimum # strings needed to write P.

Ex. $P = (6, 6, 5, 4) = (6, 6, 5) \cup (5, 4)$, so $r_P = 2$.

Thm 1.8. *(Basili [Bas2]): $Q(P)$ has r_P parts.*

Def. The *index* of a partition Q is its largest part:

So index $Q(P) = 1 + \max\{k \mid A^k \neq 0\}$.

Let $S_P =$ set of parts of P and n_i the multiplicity of i in P.

Let $s_i = \sum_{k>i} n_k$ and let $j_i = \max\{n_{i-1} + n_i, n_i + n_{i+1}\}$.

Thm 1.9 (Index of $Q(P)$). *(P. Oblak ([Ob2], later [BaI2])*

Let K be an infinite field. The index of $Q(P)$ satisfies,

$$\text{index}(Q(P)) = \max_{1\leq i\leq p_1}\{2s_i + n_i + (i-1)j_i\}. \qquad (1.1)$$

Thm 1.10. *R. Basili-I [BaI1], D. Panyushev[Pan].*

$Q(P) = P \Leftrightarrow$ the parts of P differ pairwise by ≥ 2.

(BI: P "stable" if $Q(P) = P$. Panyushev: P "self large").

The *Hilbert function* H of the commutative algebra $\mathcal{A}_{A,B} = K[A,B]$ gives the dimension in each degree of the associated graded algebra. For a codimension two algebra, the *partition* $P(H)$ is dual to the graph of H (to $\{h_i, i = 0,1,\dots\}$)

Ex. For $H = (1,2,3,3,2,2,0)$, $P(H) = (6,5,2)$.

Thm 1.11 (Pencils). *[BaI1] Suppose $A \in \mathcal{U}_B$, let $H = H(K[A,B])$ and let K be alg. closed,* char $K = 0$. *Then for generic $\lambda \in \mathbb{P}^1$ the Jordan block sizes of the action of $A + \lambda B$ on $K[A,B]$ are given by the parts of $P(H)$.*

Thm 1.12. *(T. Kosir and P. Oblak)[KO]*

$Q(Q(P)) = Q(P)$. *($Q(P)$ is "stable").*

Proof. Show $\mathcal{A} = K[A, B]$ is Gorenstein if $A \in \mathcal{N}_B$ is generic, extending a result of V. Baranovsky that $\mathcal{A}$ has a cyclic vector [Bar2001]. F.H.S. Macaulay characterized the Hilbert function $H(\mathcal{A})$, for $\mathcal{A}$ Gorenstein: H drops by at most 1 ($\forall i, h_i - h_{i+1} \leq 1$). Then the dual partition $P(H)$ is stable – that is, the parts of $P(H)$ differ pairwise by at least 2. $\square$

Ex. $H = (1, 2, 3, 4, 3, 3, 2, 2, 2, 1)$ has $P(H) = (10, 8, 4, 1)$.

Ex 1.13. $P = (3, 1)$. Take A generic in $\mathcal{N} = \mathcal{U}_B = \pi^{-1}(0, 0)$

.

$$B = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad A = \left(\begin{array}{ccc|c} 0 & a & b & f \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & d & 0 \end{array}\right).$$

$A^2 = \alpha \overrightarrow{e_{13}}, \alpha = a^2 + df$. If $\alpha \neq 0, P_A = (3, 1)$ (P is stable). When $\alpha = 0, P_A = (2, 2)$ or $(2, 1, 1)$ or $(1, 1, 1, 1)$.

Ex 1.14. $P = (3,1,1)$. $\mathcal{U}_B = \pi^{-1}\left(0, \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)\right)$

$$B = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A = \left(\begin{array}{ccc|cc} 0 & a & b & e_1 & e_2 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & f_2 & 0 & c \\ 0 & 0 & f_1 & 0 & 0 \end{array}\right) \in \mathcal{U}_B.$$

Here $A^3 = (ce_1f_1)\overrightarrow{e_{13}}, A^4 = 0$, so ($A$ generic) $Q(P) = (4,1)$.

Also, $A^3 = 0$ iff $P_A \leq (3,1,1)$,

Note: the $\mathcal{C}_B$ orbit of (3,1,1) in $\mathcal{U}_B$ is *reducible*, though its $\mathcal{C}_B$ orbit in $\mathcal{N}_B$ is *irreducible.*

We have

$$A^2 = \left(\begin{array}{ccc|cc} 0 & 0 & a^2+\beta & 0 & ce_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & cf_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A^3 = \left(\begin{array}{ccc|cc} 0 & 0 & cdf & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A^4 = 0.$$

where $\beta = e_1f_2 + e_2f_1$.

When $ce_1f_1 \neq 0$ we have $P_A = (4,1) = Q(P)$

When $ce_1f_1 = 0$ but ce_1 or cf_1 or $\alpha \neq 0$ we have rank $A^2 = 1$, and $P_A = (3,2)$ if rank$A = 3$ or $(3,1,1)$ if rank $A = 2$.

When $A^2 = 0$, $P_A = (2,2,1), (2,1,1,1)$ or $(1,1,1,1,1)$.

$$\mathcal{Q}(P) = \overline{(4,1)} = \{(4,1),(3,2),(3,1,1),(2,1,1,1),(1,1,1,1,1)\}.$$

Thm. P. Oblak [Ob1] characterizes $\mathcal{Q}(P)$ for commuting pairs (A,B) satisfying $AB = 0$.

Remark. The Hilbert scheme $\mathrm{Hilb}^n(K[x,y]/(xy))$ is a connected set of lines, and is reducible (folklore [1]). This is an example of the following result, which merits more notice:

Thm. (A. Shoshistvili [Sh], 1976). Let $\mathcal{B}$ be any local ring. Then $\mathrm{Hilb}^n(\mathcal{B})$ is $\mathbb{P}^1$ connected.

Cor. The family of commuting pairs of nilpotent matrices satisfying a finite set of polynomial conditions, and having a cyclic vector, is connected by lines $\mathbb{P}^1$.

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[1] This is easily checked, and was remarked to me by S. Kleiman and D. Eisenbud).

2 The algebra $\mathcal{U}_B$.

Let B be Jordan of partition P. Then $\mathcal{U}_B$, a maximal subalgebra of $\mathcal{N}_B$, has a filtration $U_B \supset {U_B}^2 \supset \ldots \supset {U_B}^{i(Q(P))} = 0$.

Definition 2.1. $\mathrm{Pow}(P)_{ij} = k > 0$ if both $(A^k)_{ij} \neq 0$ *and* $(A^{k+1})_{ij} = 0$ for A generic in $\mathcal{U}_B$. $\mathrm{Pow}(P)_{ij} = 0$ if $A_{ij} = 0$.

M_{X_1} = the matrix whose nonzero entries are those of A for which $\mathrm{Pow}(P)_{ij} = 1$, and whose other entries are zero.

$\mathrm{Powxe}(P) = M_{X_1} + (M_{X_1})^2 + \cdots$.

2.1 Results

i. Bases for $(U_B)^i/(U_B)^{i+1}$.

ii. An involution on $\mathcal{C}_B$ and the POS $\mathcal{D}_P$, whose restriction to $\mathcal{U}_B$ gives symmetries among the bases in (i).

iii. Algorithm to construct matrices $\mathrm{Pow}(P), \mathrm{Powxe}(P)$ closely related to the digraph $\mathcal{D}(P)$.

Ex 2.2. Let $P = (3,1,1)$, $A = \left(\begin{array}{ccc|cc} 0 & a & b & e_1 & e_2 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & f_2 & 0 & c \\ 0 & 0 & f_1 & 0 & 0 \end{array}\right)$ generic in $\mathcal{U}_B$.

$$A^2 = \left(\begin{array}{ccc|cc} 0 & 0 & a^2+\beta & 0 & e_1c \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & f_1c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad \beta = e_1f_2 + e_2f_1.$$

$A^3 = e_1f_1c^2E_{13}$.

$$\mathrm{Pow}(P) = \left(\begin{array}{ccc|cc} 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right). \quad M_{X_1}(P) = \left(\begin{array}{ccc|cc} 0 & a & 0 & e_1 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & c \\ 0 & 0 & f_1 & 0 & 0 \end{array}\right).$$

$$\mathrm{Powxe}(P) = M_{X_1}(P) + (M_{X_1}(P))^2 + \cdots = \left(\begin{array}{ccc|cc} 0 & a & ce_1f_1+a^2 & f_1 & cf_1 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & ce_1 & 0 & c \\ 0 & 0 & e_1 & 0 & 0 \end{array}\right).$$

Interpretation of Powxe(P)**,** Pow(P)**.**

Ex 2.3. Let $P = (3,1,1)$. Recall $M_{X_1}(P) = \left(\begin{array}{ccc|cc} 0 & a & 0 & f_1 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & c \\ 0 & 0 & e_1 & 0 & 0 \end{array}\right)$.

$$\text{Powxe}(P) = M_{X_1}(P) + (M_{X_1}(P))^2 + \cdots = \left(\begin{array}{ccc|cc} 0 & a & ce_1f_1 + a^2 & f_1 & cf_1 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & ce_1 & 0 & c \\ 0 & 0 & e_1 & 0 & 0 \end{array}\right).$$

Remark 2.4. The monomials in the (i,j) entry of Powxe(P) correspond to *maximal* paths from i to j, in the sense that there is no longer path from i to j properly containing the given one. Pow(P) gives the (top) degree of each entry of Powxe(P). There is an identification of edges corresponding to Toeplitz equalities in small blocks (so a set $\mathcal{T} \neq 0$).

Thm 2.5 (Basis Theorem). *A set of basis vectors for* $(\mathcal{U}_B)^k \mod \mathcal{U}_B)^{k+1}$ *are given by the vectors that correspond to the entries k of* Pow(P) *mod Hankel relations: the basis has one vector v_h for each Hankel small diagonal h whose entries equal k:* $v_h = \sum_{(i,j)\in h} e_{ij}$

A basis for $\mathcal{U}_B)^k$ *is given by those vectors as above corresponding to the small Hankel diagonals of* $\mathrm{Pow}(P)$ *whose entries are at least* k.

Ex 2.6. $P = (3, 1, 1)$.

$$A = \left(\begin{array}{ccc|cc} 0 & \underline{x_{12}} & x_{13} & \underline{x_{14}} & x_{15} \\ 0 & 0 & \underline{x_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & x_{43} & 0 & \underline{x_{45}} \\ 0 & 0 & \underline{x_{53}} & 0 & 0 \end{array}\right), \qquad \mathrm{Pow}(P) = \left(\begin{array}{ccc|cc} 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

Here $v_{11} = e_{12} + e_{23}, v_{12} = e_{14}, v_{13} = e_{45}, v_{14} = e_{53}; v_{21} = e_{15}, v_{22} = e_{43}, v_3 = e_{13}$

$U_B / {U_B}^2 = V_1 = \langle v_{11}, v_{12}, v_{13}, v_{14} \rangle$.

${U_B}^2 / {U_B}^3 = V_2 = \langle v_{21}, v_{22} \rangle$ and ${U_B}^3 = V_3 = \langle v_3 \rangle$.

The action of ι extends to V, and each V_i is ι-invariant.

Remark. There is symmetry here and for some other (not all) P in the "U_B-Hilbert functions", when stratified by large matrix blocks", corresponding to $3, (3, 1), 1$. Here $H_{U_B}(V_1) = (1, 2, 1), H_{U_B}(V_2) = (0, 2, 0)$.

Problem. Let A_i = generic element of ${U_B}^i$. We have, evidently, rank $A_i \geq$ rank A^i. Compare these ranks.

The algebra $\mathcal{U}_B$ for $B = J_P, P = (3,1,1)$.

$$B = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A = \left(\begin{array}{ccc|cc} 0 & a & b & e_1 & e_2 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & f_2 & 0 & c \\ 0 & 0 & f_1 & 0 & 0 \end{array}\right), \quad \mathrm{Pow}(P) = \left(\begin{array}{ccc|cc} 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

We give the following basis for $\mathcal{U}_B$:

$$v_{11} = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \end{array}\right) \qquad v_{21} = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 1 \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \end{array}\right)$$

$$v_{12} = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \end{array}\right) \qquad v_{22} = \left(\begin{array}{ccc|cc} \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$v_{13} = \left(\begin{array}{ccc|cc} \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad v_{3} = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \end{array}\right)$$

$$v_{14} = \left(\begin{array}{ccc|cc} \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ \multicolumn{3}{c|}{\cdots} & \multicolumn{2}{c}{\cdots} \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

Remark. We have $(\mathcal{U}_B)^3 = \langle v_3 \rangle, (\mathcal{U}_B)^2 = \langle v_{21}, v_{22}; v_3 \rangle, \mathcal{U}_B = \langle v_{11}, v_{12}, v_{13}, v_{14}; v_{21}, v_{22}; v_3 \rangle$.

The involution σ, a generalized transpose (we'll define it later) satisfies

$$\sigma(v_{12}) = v_{14}, \sigma(v_{21}) = v_{22}$$

and leaves the other basis vectors fixed.

The nonzero multiplications among the basis vectors are

$v_{13} \cdot v_{14} = v_{22}$.

$\sigma(v_{13} \cdot v_{14}) = \sigma(v_{14}) \cdot \sigma(v_{13}) = \sigma(v_{22})$, so

$v_{12} \cdot v_{13} = v_{21}$.

Also,

$v_{12} \cdot v_{22} = v_3$, and, applying σ,

$v_{21} \cdot v_{14} = v_3$.

Also $v_{11} \cdot v_{11} = v_3$.

Pairs of commuting matrices in $\mathcal{U}_B$ - an example.

Question 3. Is $\{(A, C) \mid A, B \in \mathcal{U}_B, [A, B] = 0\}$ irreducible?

(Similar question for $A, B \in \mathcal{N}_B$). *We conjecture the answer NO in general for $\mathcal{U}_B$ and $\mathcal{N}_B$.* D.I. Panyushev [2] NO: for $\mathcal{C}_B, P = (3^4, 1^2)$.

Answer for $B = J_P, P = (3, 1, 1)$ and $\mathcal{U}_B$: YES.

Use that the eigenspaces V^+ (symmetric) and V^- (antisymmetric) for σ satisfy,

$[A, B] \in V^-$ when A, B are both in V^+ or both in V^-. $(*)$

$[A, B] \in V^+$ when one of A, B in V^+ and the other in V^-. $(**)$

Since $B = v_{11}$ and $B^2 = v_3$ commute with everything in $\mathcal{U}_B$ we may reduce to v, v' each having zero component on v_{11}, v_3. Among

$c = v_{13}, w_1 = v_{12} + v_{14}, w_2 = v_{21} + v_{22} \in V^+$; and

$u_1 = v_{12} - v_{14}, u_2 = v_{21} - v_{22} \in V^-$ only the bracket $[w_1, c] = u_2$ is nonzero. Thus, for $v = a\overrightarrow{w_1} + b\,\overrightarrow{c} + c\overrightarrow{w_2} + d\overrightarrow{v_1} + e\overrightarrow{v_2}$,

and $v' = a'\overrightarrow{w_1} + b'\,\overrightarrow{c} + c'\overrightarrow{w_2} + d'\overrightarrow{v_1} + e'\overrightarrow{v_2}$ we have

$[v, v'] = (ab' - a'b)\overrightarrow{v_2}$, so $[v, v'] = 0 \Leftrightarrow ab' - a'b = 0$.

This is an irreducible condition on the coefficients of v, v' on the chosen basis!

---

[2] "Bus-ride lemma" that D.I. Panyushev showed at LAW '08. His proof uses $\mathcal{G}l_2$ action to show there is a high-dimensional component of nilpotent pairs

3 An involution on partitioned matrices

Ex 3.1. The involution $\sigma_s(2,3)$ takes $M_5(R) \to M_5(R)$:

$$\left(\begin{array}{cc|ccc} a & b & \alpha'_4 & \alpha'_5 & \alpha'_6 \\ c & d & \alpha'_1 & \alpha'_2 & \alpha'_3 \\ \hline \alpha_3 & \alpha_6 & e & f & g \\ \alpha_2 & \alpha_5 & h & i & j \\ \alpha_1 & \alpha_4 & k & l & m \end{array}\right) \text{ to } \left(\begin{array}{cc|ccc} d & b & \alpha_4 & \alpha_5 & \alpha_6 \\ c & a & \alpha_1 & \alpha_2 & \alpha_3 \\ \hline \alpha'_3 & \alpha'_6 & m & j & g \\ \alpha'_2 & \alpha'_5 & l & i & f \\ \alpha'_1 & \alpha'_4 & k & h & e \end{array}\right).$$

Definition 3.2. The action of $\sigma_s(a,b)$ on $M_{a+b}(R)$:

i. reflects the entries in the $a \times a$ block at the upper left, and in the $b \times b$ block in the lower right, about their non-main diagonals.

ii. Sends the $b \times a$ block in the lower left into the $a \times b$ block at upper right by transpose followed by reversing the order of rows, then reversing the order of columns.

3.1 An involution on the POS $\mathcal{D}'_P$ and $\mathcal{D}_P, P \to n$.

Let $P = (\ldots, i^{n_i}, \ldots)$ be a partition of $n = \sum_i i \cdot n_i$; $S_P = \{i \mid n_i \geq 1\}$.

Label the vertices $V = (1, \ldots, n)$ of the digraph:

$(i,j,k), i \in S_P, 1 \leq j \leq i, 1 \leq k \leq n_i$. We define $\sigma : V \to V$

$$\sigma(i,j,k) = (i, i+1-j, n_i+1-k). \tag{3.1}$$

On edges we define

$$\sigma\left((i,j,k),(i',j',k')\right) = \left(\sigma(i',j',k'),\sigma(i,j,k)\right). \tag{3.2}$$

Ex 3.3. [σ for $P=(3^2,1^3)$] Here $n=9$. On vertices v, we have

$$\left(\begin{array}{c|cccccc|ccc} v & 3,1,1 & 3,2,1 & 3,3,1; & 3,1,2 & 3,2,2 & 3,3,2 & 1,1,1 & 1,1,2 & 1,1,3 \\ \hline \sigma(v) & 3,3,2 & 3,2,2 & 3,1,2; & 3,3,1 & 3,2,2 & 3,1,1 & 1,1,3 & 1,1,2 & 1,1,1 \end{array}\right)$$

For $P=(3^2,1^3)$. $A \in \mathcal{A} = \mathrm{Mat}(\mathcal{D}_P)$ (take $T=\emptyset$)

$$A = \left(\begin{array}{ccc|ccc|ccc} 0 & a_1 & a_2 & d & d_2 & d_3 & f_4 & f_5 & f_6 \\ 0 & 0 & a_1' & 0 & d' & d_2' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d'' & 0 & 0 & 0 \\ \hline 0 & c & c_2 & 0 & a_3 & a_4 & f & f_2 & f_3 \\ 0 & 0 & c' & 0 & 0 & a_3' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & e_3 & 0 & 0 & e_6 & 0 & s & s_2 \\ 0 & 0 & e_2 & 0 & 0 & e_5 & 0 & 0 & t \\ 0 & 0 & e & 0 & 0 & e_4 & 0 & 0 & 0 \end{array}\right), \ \sigma(A) = \left(\begin{array}{ccc|ccc|ccc} 0 & a_3' & a_4 & d'' & d_2' & d_3 & e_4 & e_5 & e_6 \\ 0 & 0 & a_3 & 0 & d' & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\ \hline 0 & c' & c_2 & 0 & a_1' & a_2 & e & e_2 & e_3 \\ 0 & 0 & c & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & f_3 & 0 & 0 & f_6 & 0 & t & s_2 \\ 0 & 0 & f_2 & 0 & 0 & f_5 & 0 & 0 & s \\ 0 & 0 & f & 0 & 0 & f_4 & 0 & 0 & 0 \end{array}\right),$$

3.2 The involution ι for $\mathcal{C}_B$, $P=(p^a,q^b)=P_B, B$ Jordan.

Let $P=(p^a,q^b)=(p,\dots,p;q,\dots,q), p>q;\ n=ap+bq.$

Let $M = \begin{array}{ccc} & a & b \\ a & M(1,1) & M(1,2) \\ b & M(2,1) & M(2,2) \end{array}$, Form $M' = \begin{array}{ccc} & ap & bq \\ ap & M'(1,1) & M'(1,2) \\ bq & M'(2,1) & M'(2,2) \end{array}$ as follows:

Replace each entry of $M \in M_{a+b}(K)$ by small blocks, forming $M' \in M_n(K)$. The small blocks of M' have the form found in $A \in \mathcal{C}_B$:

i. The $p \times p$ small blocks of $M'(1,1)$ and $q \times q$ of $M'(2,2)$ are circulant upper triangular (c.u.t.).

ii. The $q \times p$ matrices that comprise the entries of $M(2,1)$ have the first $p-q$ columns zero, followed by a c.u.t. $q \times q$ subblock $C(2,1)_{uv}$,

$1 \leq u \leq b, 1 \leq v \leq a$.

iii. The $p \times q$ small blocks of $M(1,2)$ have the last $p-q$ rows zero, preceded by a c.u.t. $q \times q$ subblock $B(1,2)_{uv}, 1 \leq u \leq a, 1 \leq v \leq b$.

Note: We will use that circulant $q \times q$ matrices commute.

Def. For $P = (p^a, q^b)$, We define $\sigma_{s,P}$ on $\mathcal{C}_B, B = J_P$.

a. apply the involution $\sigma_s(a,b)$ to M, permuting the small blocks. However, in applying $\sigma_s(a,b)$ we must

b. replace each $q \times p$ entry $M(2,1)_{uv} = (0, C_{uv}), 1 \leq u \leq a, 1 \leq v \leq b$ of M_{21} by the $p \times q$ matrix $\begin{pmatrix} C_{uv} \\ 0 \end{pmatrix}$, and

c. replace each $p \times q$ entry $M(1,2)_{uv} = \begin{pmatrix} B_{uv} \\ 0 \end{pmatrix}$ $1 \leq u \leq b, 1 \leq v \leq a$ of M_{21} by the $q \times p$ matrix $(0, B_{uv})$.

Def. P arbitrary. Define $\sigma = \sigma_{s,P} : \mathcal{C}_B \to \mathcal{C}_B$ by defining it for each pair (p, q) of distinct elements of S_P. Since the action on the diagonal $p \times p$ blocks is independent of q, this is consistent.

Let $K[X_P]$ the ring of variables, entries of $A_{gen} \in \mathcal{C}_B$; define

$\iota : K[X_P] \to K[X_P]$ by the action of $\sigma_{s,P}$ on A_{gen}. We also define $\iota : X_1(P) \to X_1(P)$ by the action of σ on M_{X_1}.

Thm 3.4 (Involution theorem). *i. The involution σ is an anti-isomorphism on $\mathcal{C}_B$, that restricts to $\mathcal{N}_B$ and to $\mathcal{U}_B$ (that is $\sigma : \mathcal{U}_B \to \mathcal{U}_B$).*

$$\sigma(UV) = \sigma(V) \cdot \sigma(U). \tag{3.3}$$

ii. We have for $U, V \in$ subring $K[A_{gen}] \subset \mathcal{C}_B$:

$$\iota(U) = \sigma(U), \text{ and } \iota(UV) = \iota(U)\ \iota(V). \tag{3.4}$$

iii. Powxe(P) *has the symmetry*

$$\iota(\mathrm{Powxe}(P)) = \sigma(\mathrm{Powxe}(P)).$$

Ex 3.5. Let $P = (3^2, 1^3)$. Then a generic $A \in C_B$ satisfies

$$A = \left(\begin{array}{ccc|ccc|ccc}
\alpha_{11} & a_1 & a_2 & d & d_2 & d_3 & f_4 & f_5 & f_6 \\
0 & \alpha_{11} & a_1 & 0 & d & d_2 & 0 & 0 & 0 \\
0 & 0 & \alpha_{11} & 0 & 0 & d & 0 & 0 & 0 \\
\hline
\alpha_{21} & c & c_2 & \alpha_{22} & a_3 & a_4 & f & f_2 & f_3 \\
0 & \alpha_{21} & c & 0 & \alpha_{22} & a_3 & 0 & 0 & 0 \\
0 & 0 & \alpha_{21} & 0 & 0 & \alpha_{22} & 0 & 0 & 0 \\
\hline
0 & 0 & e_3 & 0 & 0 & e_6 & \beta_{11} & s & s_2 \\
0 & 0 & e_2 & 0 & 0 & e_5 & \beta_{21} & \beta_{22} & t \\
0 & 0 & e & 0 & 0 & e_4 & \beta_{31} & \beta_{32} & \beta_{33}
\end{array}\right),$$

$$\pi(A) = \left(\begin{pmatrix} \alpha_{11} & d \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \begin{pmatrix} \beta_{11} & s & s_2 \\ \beta_{21} & \beta_{22} & t \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} \right).$$

Then $\sigma_{s,P}$ reflects $\pi(A)$ about the non-main diagonals. and

$$\sigma_{s,P} : a_1 \to a_3, a_2 \to a_4; e \to f, e_i \to f_i, 2 \le i \le 6.$$

3.3 The vanishing-order matrix $\mathrm{Pow}(P)$; the matrix $\mathrm{Powxe}(P)$

Def. $X_P = \{x_{ij} \mid \text{both } A_{ij} \neq 0, A_{ij}^2 = 0, A \text{ generic in } \mathcal{U}_B\}/\text{mod}$ Hankel relations $\}$. (i.e. We identify equal circulant entries)

$M_{X_1}(P) = n \times n$ matrix with

$$M_{X_1}(P)_{ij} = \begin{cases} x_{ij} \in X_P \text{ if } A \text{ generic in } \mathcal{U}_B \text{ has entry } A_{ij} \in X_P \\ 0 \text{ otherwise.} \end{cases} \tag{3.5}$$

$\mathrm{Powxe}(P) = M_{X_1} + (M_{X_1})^2 + \cdots.$

$\mathrm{Powx}(P)_{ij} =$ highest degree term of $\mathrm{Powxe}(P)_{ij}$,

$\mathrm{Pow}(P)$ integer matrix, $\mathrm{Pow}(P)_{ij} =$ degree of $\mathrm{Powx}(P)_{ij}$.

Ex 3.6. $P = (3)$,

$$M_{X_1} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{Powxe}(P) = \begin{pmatrix} 0 & a & a^2 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$$

Ex 3.7. For $P = (3,1,1)$, recall that generic $A \in \mathcal{U}(B)$ and Powxe(P) satisfy

$$A = \left(\begin{array}{ccc|cc} 0 & \underline{a} & b & \underline{f_1} & f_2 \\ 0 & 0 & \underline{a} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & e_2 & 0 & \underline{c} \\ 0 & 0 & \underline{e_1} & 0 & 0 \end{array}\right). \tag{3.6}$$

$$\mathrm{Powxe}(P) = \left(\begin{array}{ccc|cc} 0 & a & ce_1f_1 + a^2 & f_1 & cf_1 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & ce_1 & 0 & c \\ 0 & 0 & e_1 & 0 & 0 \end{array}\right). \tag{3.7}$$

Here $\sigma : e_1 \to f_1, e_2 \to f_2$ and ι(Powxe(P) = σ(Powxe(P))).

Also $\sigma(ce_1f_1 + a^2) = ce_1f_1 + a^2$ - entry fixed by ι;

$$\text{and } \iota \text{ takes } \begin{pmatrix} ce_1 \\ e_1 \end{pmatrix} \text{ to } \begin{pmatrix} f_1 & cf_1 \end{pmatrix} = \sigma \begin{pmatrix} e_1 & ce_1 \end{pmatrix},$$

3.4 Constructing $\mathrm{Powxe}(P)$, an example.

Ex 3.8. For $P = (3^2, 1^3)$. $A \in \mathcal{U}_B$ and $\mathrm{Pow} = \mathrm{Pow}(P)$:

$$A = \left(\begin{array}{ccc|ccc|ccc}
0 & a_1 & a_2 & d & d_2 & d_3 & f_4 & f_5 & f_6 \\
0 & 0 & a_1 & 0 & d & d_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
\hline
0 & c & c_2 & 0 & a_3 & a_4 & f & f_2 & f_3 \\
0 & 0 & c & 0 & 0 & a_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & e_3 & 0 & 0 & e_6 & 0 & s & s_2 \\
0 & 0 & e_2 & 0 & 0 & e_5 & 0 & 0 & t \\
0 & 0 & e & 0 & 0 & e_4 & 0 & 0 & 0
\end{array}\right), \ \mathrm{Pow} = \left(\begin{array}{ccc|ccc|ccc}
0 & 2 & 5 & 1 & 3 & 6 & 2 & 3 & 4 \\
0 & 0 & 2 & 0 & 1 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 1 & 4 & 0 & 2 & 5 & 1 & 2 & 3 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 3 & 0 & 0 & 4 & 0 & 1 & 2 \\
0 & 0 & 2 & 0 & 0 & 3 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0
\end{array}\right).$$

Here the variables X_1 of M_{X_1} are $\{c, d, e, f, s, t\}$ and correspond to the entries 1 of $\mathrm{Pow}(P)$.

We have for $P = (3^2, 1^3) = (3,3,1,1,1), \mathrm{Powxe}(P)$ is

$$\left(\begin{array}{ccc|ccc|ccc}
0 & cd & defst + c^2d^2 & d & cd^2 & \underline{d^2efst} + c^2d^3 & df & dfs & dfst \\
0 & 0 & cd & 0 & d & cd^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 \\
\hline
0 & c & efst + c^2d & 0 & cd & defst + c^2d^2 & f & fs & fst \\
0 & 0 & c & 0 & 0 & cd & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & est & 0 & 0 & dest & 0 & s & st \\
0 & 0 & et & 0 & 0 & det & 0 & 0 & t \\
0 & 0 & e & 0 & 0 & de & 0 & 0 & 0
\end{array}\right).$$

Here $Q(P)$ has two parts (by an R. Basili result, as $P = p^a, q^b, p > q+1$ has $r_P = 2$); the highest nonzero power of a generic $A \in \mathcal{U}_B$ is $A^6 = \underline{d^2efst}\, E_{16}$, hence $Q(P) = (7,2)$.

Here Powxe(P) shows the symmety

$$\iota(\mathrm{Powxe}(P)) = \sigma(\mathrm{Powxe}(P),$$

and is evidently simply constructed from $M_{X_1} \in \mathcal{U}_B$.

We outline without detail the following result [BaI2, Theorem 3.27].

Thm 3.9 (Algorithm for constructing $\mathrm{Powxe}(P)$**). .**

i. Begin with M_{X_1} *(simply defined).*

ii. The diagonal blocks for each $p \in S_P$ *are the same. All other* $p \times p$ *blocks are simply constructed from them.*

iii. Diagonal $p \times p$ *blocks determine terms of* $q \times p$ *blocks,* $q > p$*, acting via the* $p \times q$ *blocks.*

iv. There is a weaker influence of larger diagonal blocks on smaller ones, when $2q \geq p \geq q$*.*

v. Begin with the smallest diagonal blocks, and construct $\mathrm{Powxe}(P)$ *in stages,*

A second algorithm [BaI2, Theorem 3.32] constructs $\mathrm{Powxe}(P)$ by induction on the order of the nonzero entries of each block, using a notion of star product of first lines of Hankel matrices.

3.5 $\mathrm{Pow}(P)$ and a basis for ${\mathcal{U}_B}^i$.

[3] Let $P = (p_1^{r_1}, \dots, p_t^{r_t}), p_1 > \cdots > p_t$, and let A be a generic element of $\mathcal{U}_B$. If the entry $A_{ij} \neq 0$ and $A_{ij} \neq A_{i-1,j-1}$ we denote it by x_{ij}, and the set of all such by X_P (one variable for each small Hankel diagonal). Let $s_i = r_1 + \cdots + r_{i+1}$. Considering $\pi : \mathcal{C}_B \to \mathcal{C}'_B$, $\dim_K(\mathcal{U}_B) = \# X_P$ satisfies

$$\# X_P = \sum_i \left(i r_i \left(r_i + 2s_i \right) - r_i \left(\frac{r_i + 1}{2} \right) \right). \tag{3.8}$$

Let $S_P = \{i \mid r_i > 0\}$, and $\forall i \in S_P$, $j_i = r_i + \max\{r_{i-1}, r_{i+1}\}$ (jump index), $s = \sum r_i$, and recall $t = \# S_P$. We denote by

$$X_k = \{x_{ij} \in X_P \mid A_{ij}^k \neq 0 \text{ but } A_{ij}^{k+1} = 0\} \tag{3.9}$$

Thus, X_k comprise the distinct variables from X_P corresponding to entries k of $\mathrm{Pow}(P)$. We have [BaI2, Sec. 3.1]

$$\# X_1 = s + 2(t-1) - \# \{i \mid j_i > r_i\} \tag{3.10}$$

---

[3] This section, an algebraic interpretation of some of the results in [BaI2], was inspired by our discussions at the 'CA meets AC' conference January 08 with J. Weyman and T. Košir.

We let $\mathcal{B}_P = I + \mathcal{U}_B$, and filter it by the ideals

$$\mathcal{B}_P \supset U_B \supset {U_B}^2 \supset \cdots \supset {U_B}^{e_P} \supset 0.$$

Here $e_P = i(Q(P)) - 1, i(Q(P)) =$ index of $Q(P)$, the largest part. We set $U_B^0 = \mathcal{B}_P$. Denote by $E = \langle \{e_{ij}, 1 \le i, j \le n\} \rangle$, the n^2-dim vector space. For $x_{ij} \in X_P$, let $v_{ij} \in E$ satisfy $v_{ij} = \sum' e_{uv}$ where $\sum'$ is over $\{uv \mid A_{uv} = x_{ij}\}$. Let $V_k = \{v_{ij}, \mid x_{ij} \in X_k\}$, and $\langle V_k \rangle \subset E$ their span, $V = \sum_{k=1}^{e_P} V_k$.

Thm 3.10. *We have the internal direct sums*

A. $\mathcal{B}_P = \oplus_{k=0}^{e_P} \langle V_i \rangle \cong \oplus_{k=0}^{e_P} {U_B}^k / {U_B}^{k+1}$*;*

B. for $i \ge 0$, $(U_B)^i = \oplus_{k \ge i} \langle V_k \rangle$*.*

Proof Outline. We write e_{ij} also for the corresponding element of U_B, provided $x_{ij} \in X_P$. (So $U_B \subset V$). Let $u \in {U_B}^k \subset E$ have nonzero component on some e_{ij} (with $x_{ij} \in X_k$). Then we achieve v_{ij} as a product of k elements $v_1 \times \cdots \times v_k, v_i \in V_1$.

□

Ex 3.11. (Repeat of Example 2.6). $P = (3, 1, 1)$.

$$A = \left(\begin{array}{ccc|cc} 0 & \underline{x_{12}} & x_{13} & \underline{x_{14}} & x_{15} \\ 0 & 0 & \underline{x_{12}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & x_{43} & 0 & \underline{x_{45}} \\ 0 & 0 & \underline{x_{53}} & 0 & 0 \end{array}\right).$$

Here $v_{11} = e_{12} + e_{23}, v_{12} = e_{14}, v_{13} = e_{45}, v_{14} = e_{53}; v_{21} = e_{15}, v_{22} = e_{43}, v_3 = e_{13}$

$U_B/{U_B}^2 = V_1 = \langle v_{11}, v_{12}, v_{13}, v_{14} \rangle$.

${U_B}^2/{U_B}^3 = V_2 = \langle v_{21}, v_{22} \rangle$ and ${U_B}^3 = V_3 = \langle v_3 \rangle$.

The action of ι extends to V, and each V_i is ι-invariant.

Remark. There is symmetry here and for some other (not all) P in the "U_B-Hilbert functions", when stratified by large matrix blocks", corresponding to $3, (3, 1), 1$. Here $H_{U_B}(V_1) = (1, 2, 1), H_{U_B}(V_2) = (0, 2, 0)$.

Problem. Let $A_i =$ generic element of ${U_B}^i$. We have, evidently, rank $A_i \geq$ rank A^i. Compare these ranks.

4 What is $Q_S(P)$ – maximal nilpotent orbit in $\pi^{-1}(M_S(B))$?[4]

4.1 Nilpotent multi-orbits $M_S(B) \subset M(B)$.

Definition 4.1. Let $P = (p_1^{r_1}, \dots, p_k^{r_k}), p_1 > \dots > p_k$. Let $\langle r_i \rangle$ = POS of partitions of r_i. Let $S = (S_1, \dots, S_k), S_i \in r_i, 1 \leq i \leq k$. Let $\mathfrak{S}(P) = \{S \in \langle r_1 \rangle \times \cdots \times \langle r_k \rangle\}$.

$M_S(B)$ = nilpotent multi-orbit in $M_{r_1}(K) \times \cdots \times M_{r_k}(K)$ determined by S.

Since $M_S(B)$ is irreducible and $\pi^{-1}(M_S(B))$ is fibred over $M_S(B)$ by an affine space isomorphic to the Jacobson radical $\mathfrak{J}$ of $\mathcal{C}_B$, we have $\pi^{-1}(M_S(B))$ is irreducible.

We denote by $Q_S(B)$ the partition giving the Jordan blocks of a generic element of $\pi^{-1}(M_S(B))$.

Ex 4.2. When $S = ((r_1), \dots, (r_k))$ (each S_i a single Jordan block), then $M_S(B) = M(B), Q_S(B) = Q(B)$.

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[4] This section was not given at LAW '08, and is from the talk at Dalhousie, Jan. 08.

Let $0 = S_0 = ((1^{r_1}), \ldots, (1^{r_k}))$ then $M_S(B) = \{(0, \ldots, 0)\}$, and $Q_0(B)$ is the maximal partition for an element of $\mathfrak{J}$.

Observation. When the distinct parts of P differ by two or more, then $Q_0(P) = P$; otherwise, $Q_0(P) \neq P$.

For $P = (2, 1^3), S = ((1), (1^3))$, then $Q_0(B) = (3, 1, 1) \neq P$.

Problem: Find $Q_S(B)$ for each S. Interpolates between $Q(P)$, and the generic orbit for $\mathcal{A} \in \mathfrak{J}$, the Jacobson radical.

Lem 4.3 (Lifting). *i. Let $\sigma \in Gl_{r_1}(K) \times \cdots \times Gl_{r_k}(K)$ and $M, M' \in M(B)$, and let $A \in \mathcal{C}_B$ with $\pi(A) = M$. Then there is a unit $\sigma' \in \mathcal{C}_B$ such that $\pi(\sigma'(A)) = A'$.*

ii. $Q_S(P) = P_A$ for A generic in $\pi^{-1}(J_{S_1}, \ldots, J_{S_k})$.

That is, in finding $Q_S(P)$ we may assume that $\pi(A)$ has components each in Jordan block form.

4.2 The partition $Q_S(P)$

Def: For a fixed P denote by $\mathfrak{Q}(P)$ the POS

$$\mathfrak{Q}(P) = \{Q_S(P) \quad \forall S \in (\mathfrak{P}(r_1) \times \cdots \times \mathfrak{P}(r_k))\},$$

Lem 4.4. *: $S \to Q_S(P)$ is a map of POS: $\mathfrak{S}(P) \to \mathfrak{Q}(P)$.*

For a partition $(S_1 = (s_{11}, \ldots, s_{1t})$, we let $m(S_1) = (ms_{11}, \ldots ms_{1t})$.

Ex 4.5 (Observation). Let $P = (m^a) = (m, \ldots, m)$, and let S_1 be a partition of (a). Then $Q_{S_1}(P) = m(S_1)$.

Ex 4.6 (Observation). [$Q_S(P)$ for hooks] Let $P = (p, 1^b) \mid p > 1$. Then the map $S \to Q_S(P) : \mathfrak{S} \to \mathfrak{Q}(P)$, is an isomorphism of lattices.

$Q_0(P) = P$ if $p \geq 3$; $\quad Q_0(P) = (3, 1^{b-1})$ if $p = 2$.

Let $S = ((1), R), T \in \mathcal{P}_B$. Then $Q_S(P)$ is obtained by "adding" T to $Q_0(P)$: add $T_i - 1$ to $Q_0(P)_i, i = 1, 2, \ldots$ until the sum n is attained.

Ex $P = (2, 1^4)$ (see Ex 3.7B). $Q_0(P) = (3, 1, 1)$. $S = (2, 2)$

$$Q_S(P) = (2, 2) + (3, 1, 1, 1) = (3 + 2 - 1, 1 + 2 - 1) = (4, 2)$$

Ex 4.7. Hooks, $p = 2$.

A. $P = (2, 1^3)$; $\mathfrak{S} = \langle 1 \rangle \times \langle 3 \rangle$.

$Q_S(P)$	S
(5)	(3)
$(4,1)$	$(2,1)$
$(3,1,1)$	$(1,1,1)$

B. $P = (2, 1^4)$; $\mathfrak{S} = \langle 1 \rangle \times \langle 4 \rangle$.

$Q_S(P)$	S
(6)	(4)
$(5,1)$ $\qquad$ $(4,2)$	$(3,1)$ $\qquad$ $(2,2)$
$(4,1,1)$	$(2,1,1)$
$(3,1^3)$	(1^4)

Ex 4.8. Hook: $p = 3$. $P = (3, 1^4)$; $\mathfrak{S} = \langle 1 \rangle \times \langle 4 \rangle$.

$$\begin{array}{ccc}
Q_S(P) & & S \\
(6,1) & \longleftarrow & (4) \\
(5,1,1) \qquad (4,2,1) & \longleftarrow & (3,1) \qquad (2,2) \\
(4,1,1,1) & \longleftarrow & (2,1,1) \\
(3,1^4) & \longleftarrow & (1^4)
\end{array}$$

Ex 4.9. $P = (2^2, 1^3)$; $\mathfrak{S} = \langle 2 \rangle \times \langle 3 \rangle$.

$$\begin{array}{lcl}
Q_S(P) & & S \\
(7) & \longleftarrow & (2) \times (3) \\
(5,2) & \longleftarrow & (1,1) \times (3) \qquad (2) \times (2,1) \\
(4,3) & \longleftarrow & (1,1) \times (2,1) \\
(4,2,1) & \longleftarrow & (2) \times (1,1,1) \\
(3,3,1) & \longleftarrow & (1,1) \times (1,1,1)
\end{array}$$

$\mathfrak{S}(P) \to \mathfrak{Q}(P)$ is *not* an isomorphism of POS.

($(1,1) \times (2,1)$ and $(2) \times (1,1,1)$ are incomparable in $\mathfrak{S}(P)$.)

4.3 Questions: the involution ι and $Q_S(P)$.

a. To what extent is $Q_S(P)$ an invariant of the digraph $\mathcal{D}(A)$, or digraph wih involution ι, for A generic in $U_S(B)$?

b. What other invariants of P are steps toward $Q_S(P)$?

c. Fix P. The condition of A being in $\pi^{-1}(J_{r_1}, \ldots, J_{r_k})$ leads to a different digraph-with-involution $\mathcal{D}'$ than $\mathcal{D}$ for A generic in $\mathcal{U}_B$. But the lengths of longest paths from $i \to j$ are unchanged, as the matrix M_{X_1} is in this fibre.

   Is the S. Poljak calculation of partitions for the generic matrices of digraphs $\mathcal{D}, \mathcal{D}'$ the same? And what is their relation to $Q(P)$?

d. Can the ranks of A^k, A generic in $\mathcal{U}_B$ be concluded from those of certain powers (or powers and sums) of M_{X_1}?

e. Fix $S = (S_1, \ldots, S_k)$. By regarding the intersection of $X_1(P)$ with $\pi^{-1}(J_{S_1}, \ldots, J_{S_k})$, one can construct variables

$X_1(S)$ and matrices $M_{X_1(S)}$. Can the ranks of powers of generic elements of the same fibres, be figured from the ranks of powers and sums of $M_{X_1(S)}$?

f. Work in the projectification of $\mathcal{C}_B, \mathcal{N}_B$, and $\mathcal{U}_B$. What are the dimensions, closures, and variety structure (CM, irreducible components, types of singularities) of various subvarieties, in particular of the orbits under conjugation by $\mathcal{C}_B^*$.

g. What are the intersections of closures of orbits? Relate this problem to analogous problems on the Hilbert scheme.

h. Consider a Ellingsrud-Strömme approach to $\mathcal{N}_B$: what can one say about homology classes of various subsets, the fixed points of $\mathbb{C}^*$, torus actions, and the related cellular decomposition? Fix the Hilbert function H. This relates also to the cells G_H, Z_H (graded, or general quotients of $k[x, y]$ having Hilbert function H).

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