An Aperiodic Tiling from a Dynamical System: An Exposition of An Example of Culik and Kari

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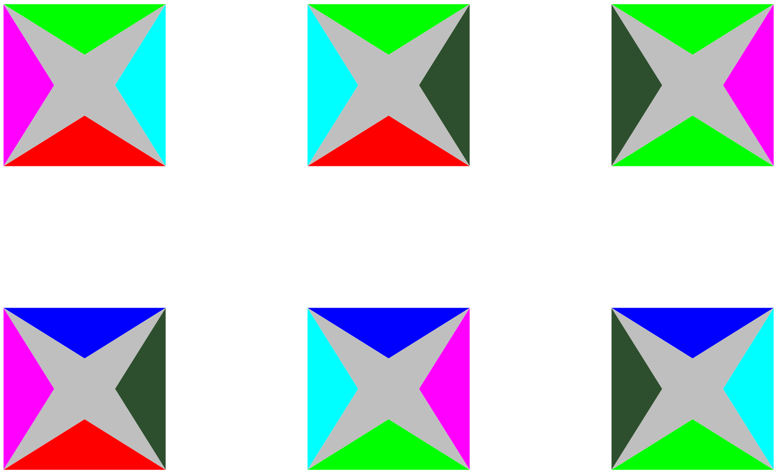
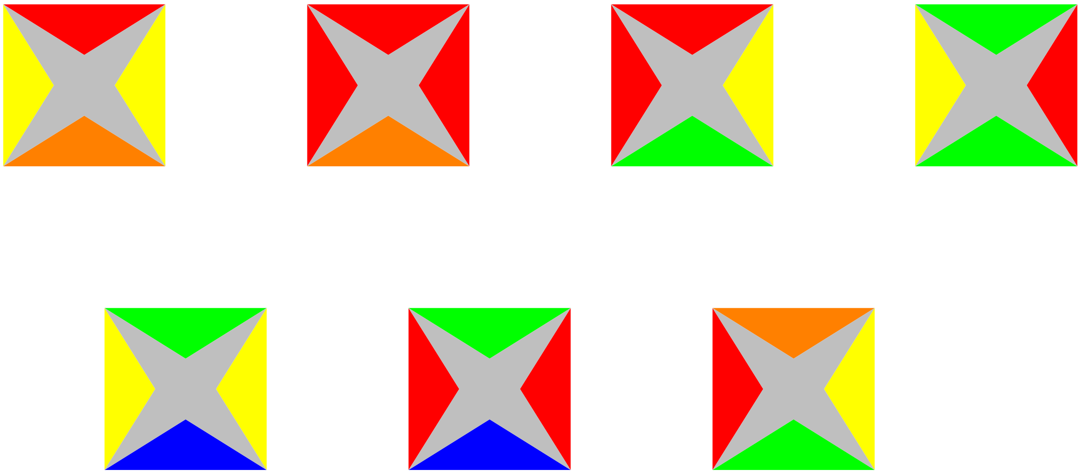
These 13 tiles can tile the plane
But only Aperiodically

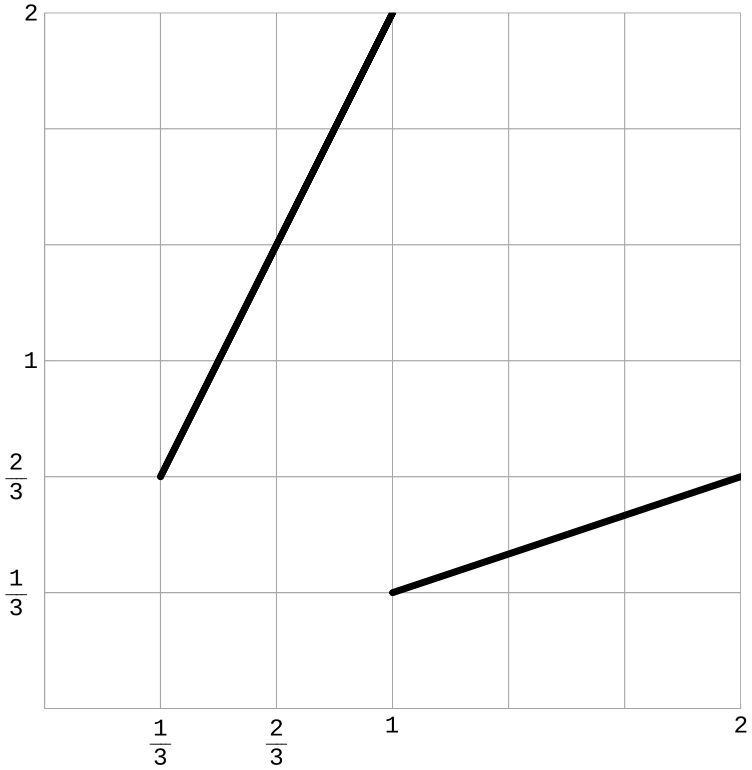




Example A (Culik-Kari)

Dynamical System A is given by the function f defined on the interval $[\frac{1}{3}, 2)$.

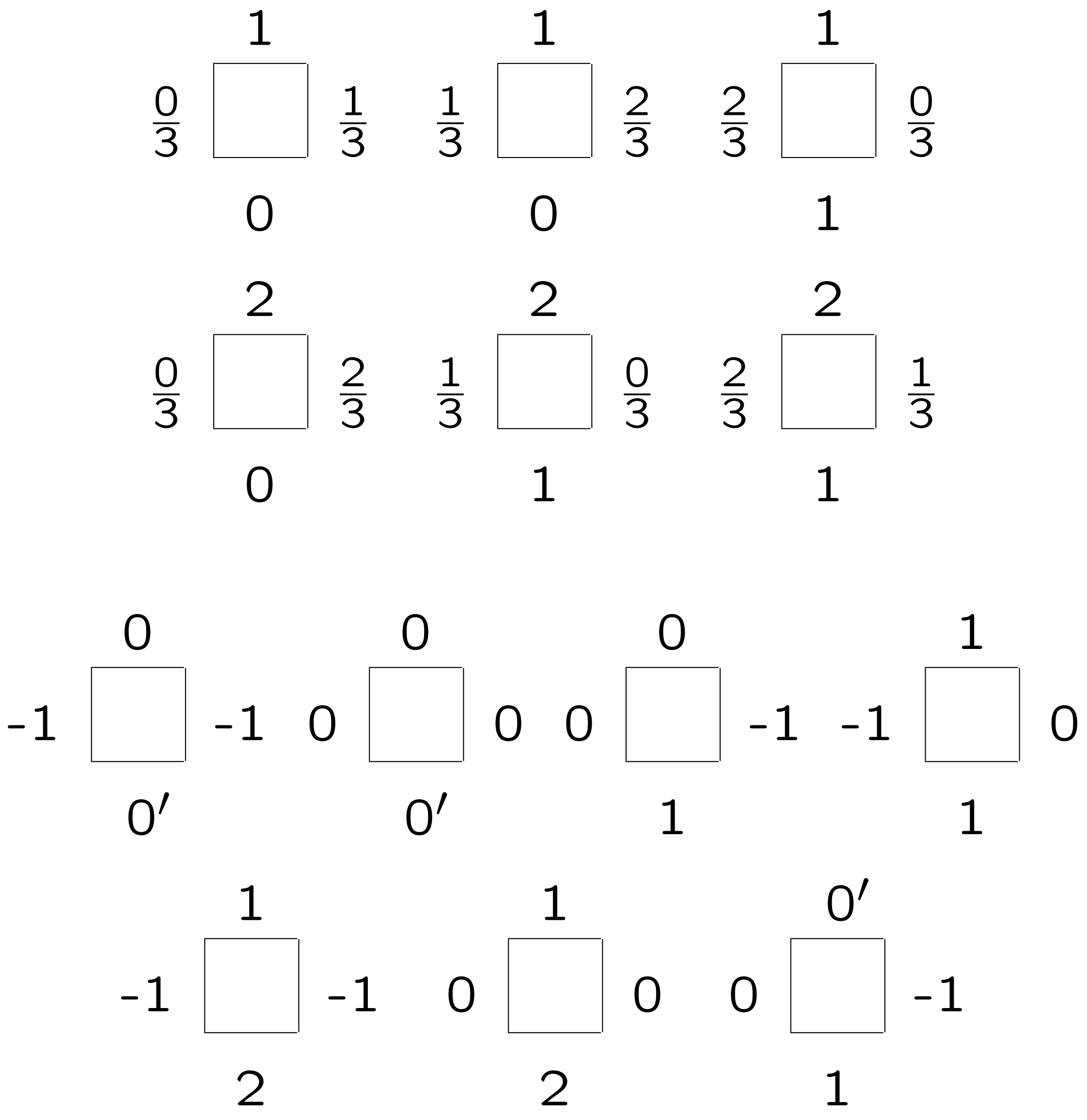
$$f(x) = \begin{cases} 2x, & \frac{1}{3} \leq x < 1 \\ \frac{1}{3}x, & 1 \leq x < 2 \end{cases}$$



$$Illustration: \frac{3}{5} \rightarrow \frac{6}{5} \rightarrow \frac{2}{5} \rightarrow \frac{4}{5}$$

Tiling Set for Example A

From $f(x)$, get a set $\mathcal{T}$ of 13 Wang Tiles.



Wang Tiles

Wang tiles are unit square tiles with **colored edges**.

In Example A, numbers color the edges.

Note 0, $0'$ and $\frac{0}{3}$ are considered different colors, but are all 0 numerically.

A **tiling set** $\mathcal{T}$ is a collection of finitely many Wang tiles $T \in \mathcal{T}$, each of which may be copied as much as needed.

The tiles are placed edge-to-edge with common edges having matching colors (or numbers in this case).

Rotations and flips (reflections) are not permitted.

A tiling set which can tile the plane is said to have a **valid tiling**.

Dynamical Systems

Finite Wang tile sets can be obtained from a wide class functions.

A piecewise, rationally multiplier (invertible) function

$$g(x) = \begin{cases} q_1 x, & x_0 \le x < x_1 \\ q_2 x, & x_1 \le x < x_2 \\ & \vdots \\ q_k x, & x_{k-1} \le x < x_k \end{cases}$$

is given by k positive, rational numbers $\{q_1, \cdots, q_k\}$ and a finite interval $[x_0, x_k)$ divided into k subintervals given by $0 \le x_0 < x_1 < \cdots < x_k$ with the q_i and x_j chosen so that g is (one-to-one), onto (and invertible).

One then asks how various properties of the dynamical system are reflected in the tilings of the plane.

Summarizing Construction Results

From **Basic Tile Construction**:

- Two-sided orbits of $g(x)$ will give tilings of the plane.

- Each row corresponds to the Beatty Expansion of x.

- The row for $g(x)$ is below the row for x.

From **Color Tweaking** $0 \neq 0'$:

- Aperiodic points for $g(x)$ will give rise to Aperiodic tilings of the plane. Periodic points give periodic tilings.

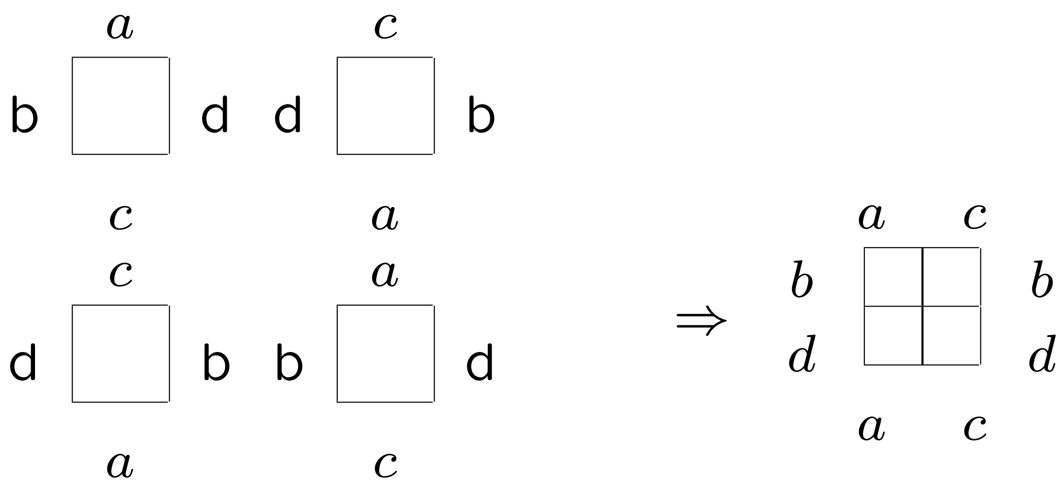
- If g has no periodic points, the tile set will have no periodic tilings.

Example: Periodic/Rotation

Any tile and its 180^o rotation tiles the plane.

Label the four colors a, b, c, d, (some may be the same).

Make a two-by-block as follows.

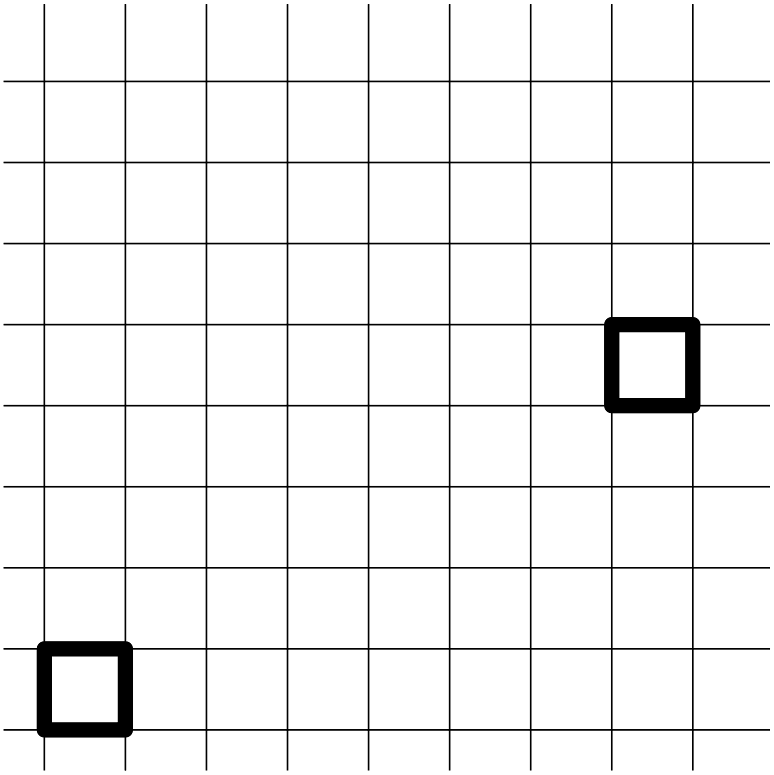


The two-by-two block has the same colors on the top and bottom, and the same colors on the left and right.

Hence it can be repeated periodically.

Periodicity

A valid tiling is **periodic** with period $(h, v) \in Z^2 - \{0, 0\}$ if the tile at position (i, j) is the same as the tile at position $(i+h, j+v)$ for all $(i, j) \in Z^2$.



Same Tile Translated

The Rotation example has period $(2,2)$. (and also $(1,1),(2,0),(0,2)$)

A tile set may have more than one valid tiling; some of which may be periodic and some of which may not.

A tile set is called **aperiodic** if it has at least one valid tiling, but does not have a periodic valid tiling.

The tile set in Example A is aperiodic.

Some History

Hao Wang was interested in automatic theorem proving.

Wang tiles are of theoretical importance because any Turing machine can be mimiced by some set of Wang tiles.

As a curious aside, there is a set of 30 tiles which have a unique tiling of the quarter plane - in which two special tiles P and C occur only at prime and composite locations along the positive x-axis.

Wang [13] conjectured in 1961 that if a set of tiles can tile the plane - then it can also tile the plane periodically.

His bigger question was whether there existed an algorithm which could determine if any given set of tiles could tile the plane.

In 1966 R. Berger showed: There is no algorithm which can Decide for all tile sets $\mathcal{T}$, whether or not $\mathcal{T}$ can tile the plane.

That is, every algorithm must fail on for some tile set (either it gives the wrong answer, or it never stops).

A corollary to Berger's theorem is that there must exist an aperiodic tile set.

In proving his theorem Berger constructed an example having 20,426 tiles.

The size of aperiodic sets has been going down ever since.

Penrose's Kites and Darts can be used to make an aperiodic set of 16 Wang tiles.

In 1995, J. Kari [6] and K. Culik [3] constructed sets of 14 and 13 tiles respectively that tile only aperiodically.

Open Problem: determine $W > 0$ such that any set $\mathcal{T}$ of size $w \leq W$ which has a valid tiling must also have a periodic tiling. As far as we know $4^* \leq W < 13$.

* Lost Theorem of Robinson

Rectangular Tilings

In the Rotation Example given earlier, the constructed two-by-two block extends to a valid tiling which has two linearly independent periods $(2,0)$ and $(0,2)$.

A **Rectangular** tiling is a valid tiling which has two periods $(n,0)$, $(0,m)$, $n,m>0$.

Having a rectangular tiling is not stronger than having a periodic tiling.

Proposition If a set of tiles admits a periodic tiling of the plane, then it also admits a rectangular tiling.

The One Dimensional result is: If a set of tiles (intervals) has a valid tiling of the real line, then it has a perodic tiling.

Higher Dimensional Result: If a set of n-dimensional Wang cubes have a valid tiling of n-dimensional space and this tiling has $n-1$ linearly independent periods then the tile set has an n-dimensional rectangular tiling.

To Do

There are three things to show.

1. The tiles can tile the plane

2. The tiles cannot tile periodically

3. How the tiles are derived from the dynamical system.

Aperiodicity of Example A

The aperiodicity of the tile set in Example A follows the same reasoning as the proof that f has no periodic points.

Lemma 1 *The dynamical system f in Example A has no periodic points.*

Proof. Suppose $f^n(x) = x$ for $n > 0$.

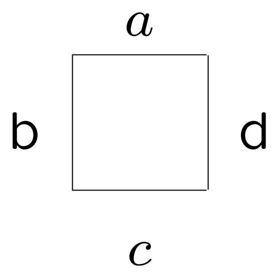
$\Rightarrow f^n(x) = q_n \cdot q_{n-1} \cdots q_1 \cdot x$ where $q_i \in \{\frac{1}{3}, 2\}$.

$\Rightarrow f^n(x) = \frac{2^{n-k}}{3^k} \cdot x = x$ for some $0 \leq k \leq n$.

Dividing by $x \in [\frac{1}{3}, 2)$ gives $\frac{2^{n-k}}{3^k} = 1$ a contradiction.

To understand how this apply to the tiles we need the definition of a multiplier tile.

Multiplier Tiles



is a **multiplier tile** with **multiplier** q if

$$q \cdot a + b - d = c$$

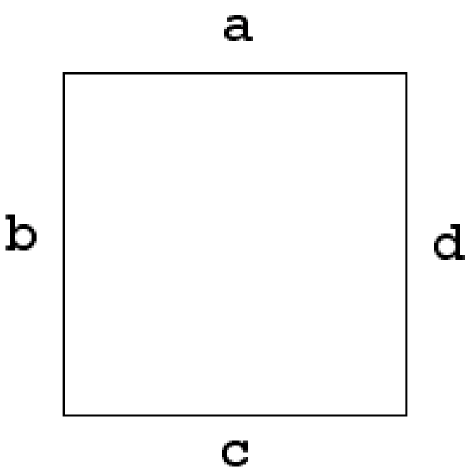
The multiplier for a tile is unique if $a \neq 0$.

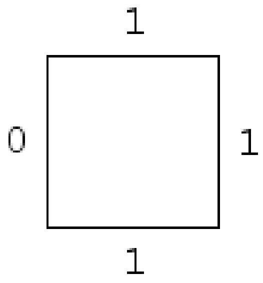
If $a = 0$ then the multiplier need not be unique.

- First 6 tiles in Example A have multiplier $\frac{1}{3}$.
- Last 7 tiles in Example A have multiplier 2.

Tile: Multiples by q

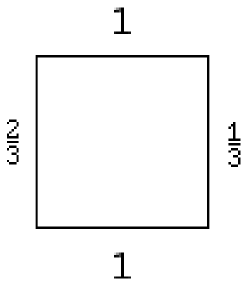
$$qa + b - d = c$$





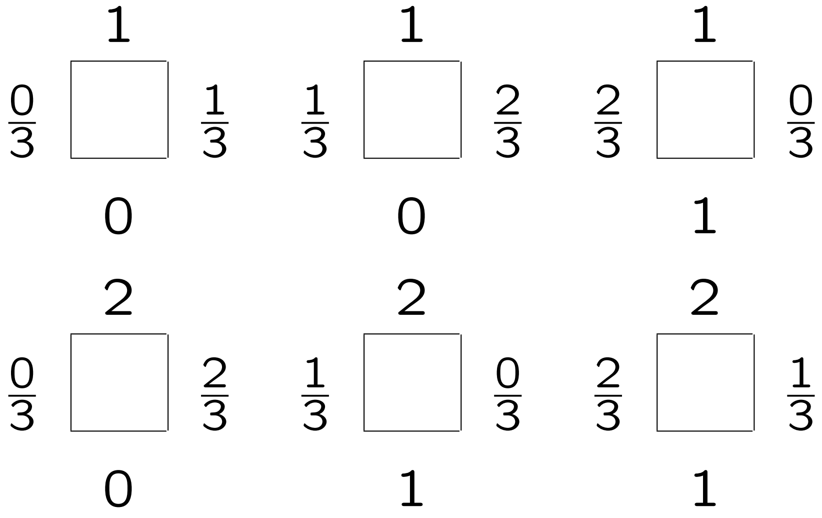
$$\mathbf{2} \cdot 1 + 0 - 1 = 1$$

Multiplies by 2



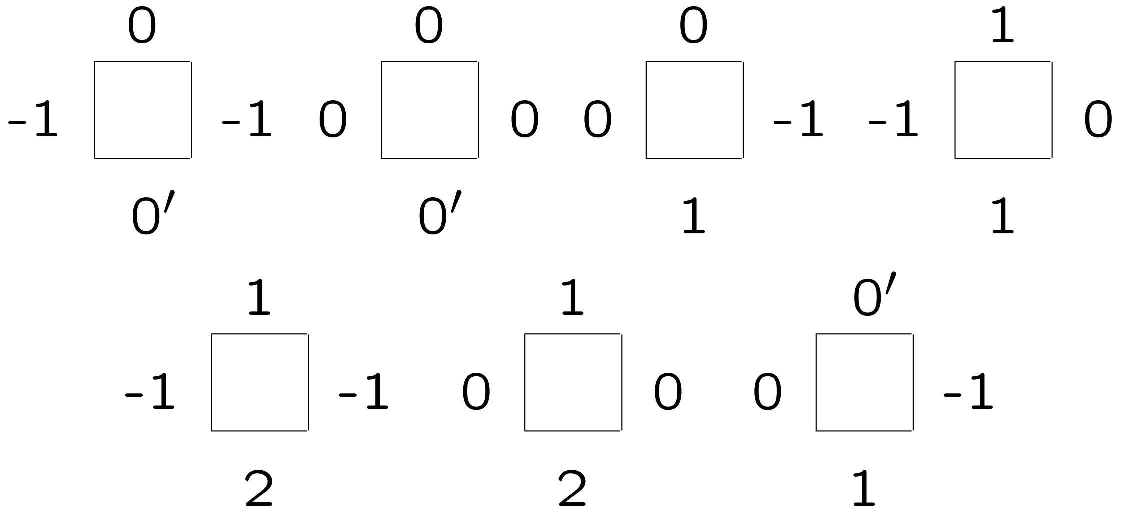
$$\frac{\mathbf{2}}{\mathbf{3}} \cdot 1 + \frac{2}{3} - \frac{1}{3} = 1$$

Multiplies by 2/3



Multiplier $\frac{1}{3}$ - Side Colors $\frac{0}{3}, \frac{1}{3}, \frac{2}{3}$

Tile Set $\frac{1}{3}$



Multiplier 2 - Side Colors $\{0, 1\}$

Tile Set $2'$

More Facts about Tile Set A

- Side colors of the Sets $\frac{1}{3}$ and $2'$ are distinct.

- In a valid tiling of the plane, each row consists of tiles Solely from set $\frac{1}{3}$ or Solely from set $2'$. $(0 \neq \frac{0}{3})$

Additional analysis *of the tiles* yields

- If a row consists of tiles from Set $\frac{1}{3}$ then the row immediately below it contains only tiles from Set $2'$.

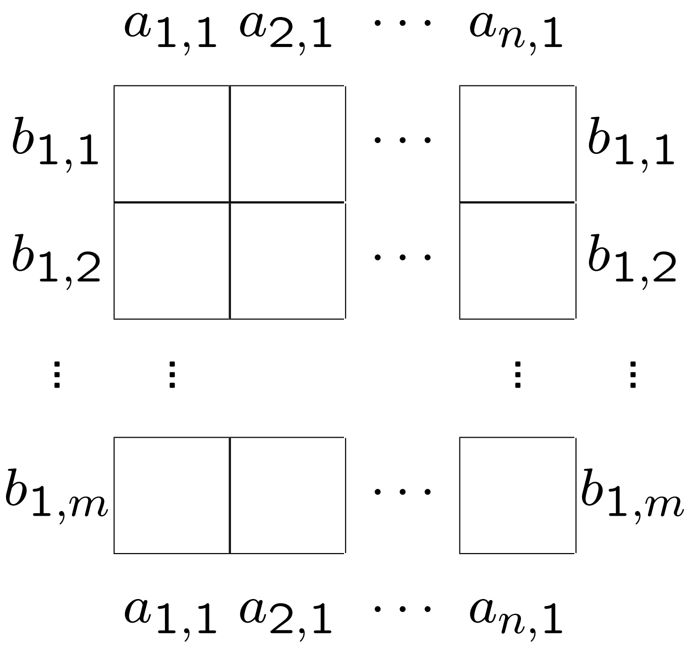
- There can be at most two consecutive rows of tiles from Set $2'$.

- Tiles from both Set $\frac{1}{3}$ and Set $2'$ must appear in any tiling of the plane.

These ref

Theorem 2 *The tile set in Example A does not possess a period.*

Assume we have a periodic block



The Multiplier rule for individual blocks becomes

$$q_m \cdots q_2 q_1 \sum_{i=1}^{n} a_{i,1} = \sum_{i=1}^{n} a_{i,1}$$

Recall that at least one of these rows consists of tiles from Set $\frac{1}{3}$.

By periodicity, assume it is the top row - which have top colors $\{1, 2\}$.

Divide by $\sum_{i=1}^{n} a_{i,1} \neq 0$ getting $\prod_{j=1}^{m} q_j = 1$.

As before $q_j \in \{2, \frac{1}{3}\}$, and we conclude that no periodicity can occur.

Existence of a Valid Tiling

We need to show there exists a valid tiling.

This will follow directly from the construction of the tiles from the dynamical system.

Step 1: Show the general method of constructing tiles from a dynamical system.

This will not give the aperiodicity result.

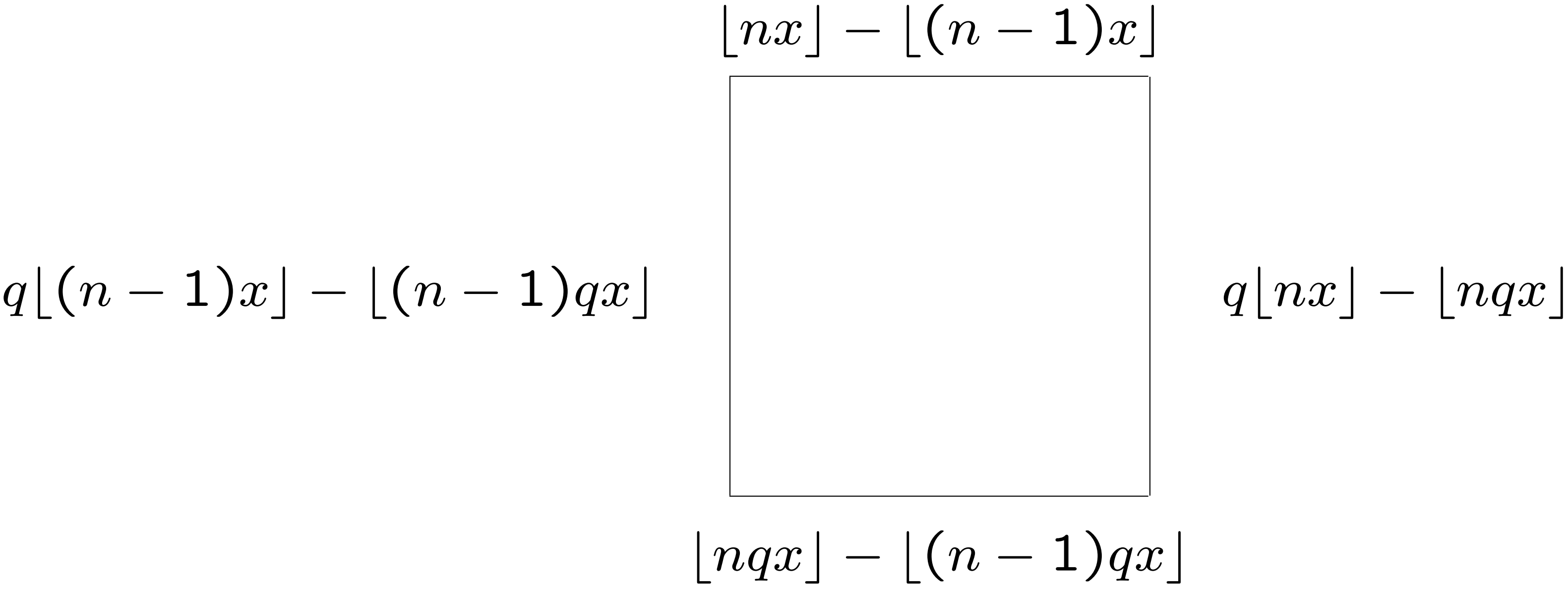
Step 2: How to "tweak" the tile set to get the aperiodicity.

Essentially this is where the different colors $0, 0', \frac{0}{3}$ come in.

Recall: We have already seen the reason $\frac{0}{3}$ is not the same color as 0. These are "side" Tweakings.

Basic Tile Construction: $B(q,x,n)$

All tiles in Example A are constructed (numerically) in the following form.



$\lfloor nx\rfloor - \lfloor (n-1)x\rfloor$

$q\lfloor (n-1)x\rfloor - \lfloor (n-1)qx\rfloor$

$q\lfloor nx\rfloor - \lfloor nqx\rfloor$

$\lfloor nqx\rfloor - \lfloor (n-1)qx\rfloor$

Here, $x > 0$ is a real number,

$q > 0$ is a rational,

n is an integer

$\lfloor x\rfloor$ is the greatest integer function.

A straightforward calculation gives

- The Basic Tile is a multiplier tile with multiplier q.

Recall $\begin{array}{ccc} & a & \\ \text{b} & \boxed{} & \text{d} \\ & c & \end{array}$ has multiplier q: $q\cdot a+b-d=c$

For the Basic Tile we have

$$\begin{aligned} & q\ (\lfloor nx\rfloor-\lfloor (n-1)x\rfloor) \\ + & \ (q\lfloor (n-1)x\rfloor-\lfloor (n-1)qx\rfloor) \\ - & \ (q\lfloor nx\rfloor-\lfloor nqx\rfloor) \quad = \lfloor nqx\rfloor-\lfloor (n-1)qx\rfloor \end{aligned}$$

A Finite Number of Tiles

Theorem 3 *If*

q *a fixed rational*

$[a, b)$ *a finite interval*

Then there are only a finite number of tiles $B(q, x, n)$*,* $x \in [a, b + 1)$*.*

To prove this, simply show that the four sides of the Basic Tile can assume only a finite number of values.

This is just a lot of straightforward calculations.

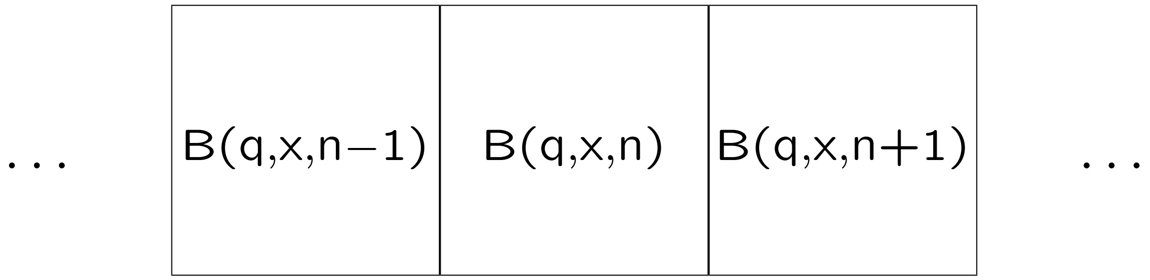
Orbits → Tilings

Fix a point x with an infinite Two-sided orbit.

Assume $f(x) = q \cdot x$.

Construct Tiles $B(q, x, n)$ for $-\infty < n < \infty$.

- These Fit together to form a row.

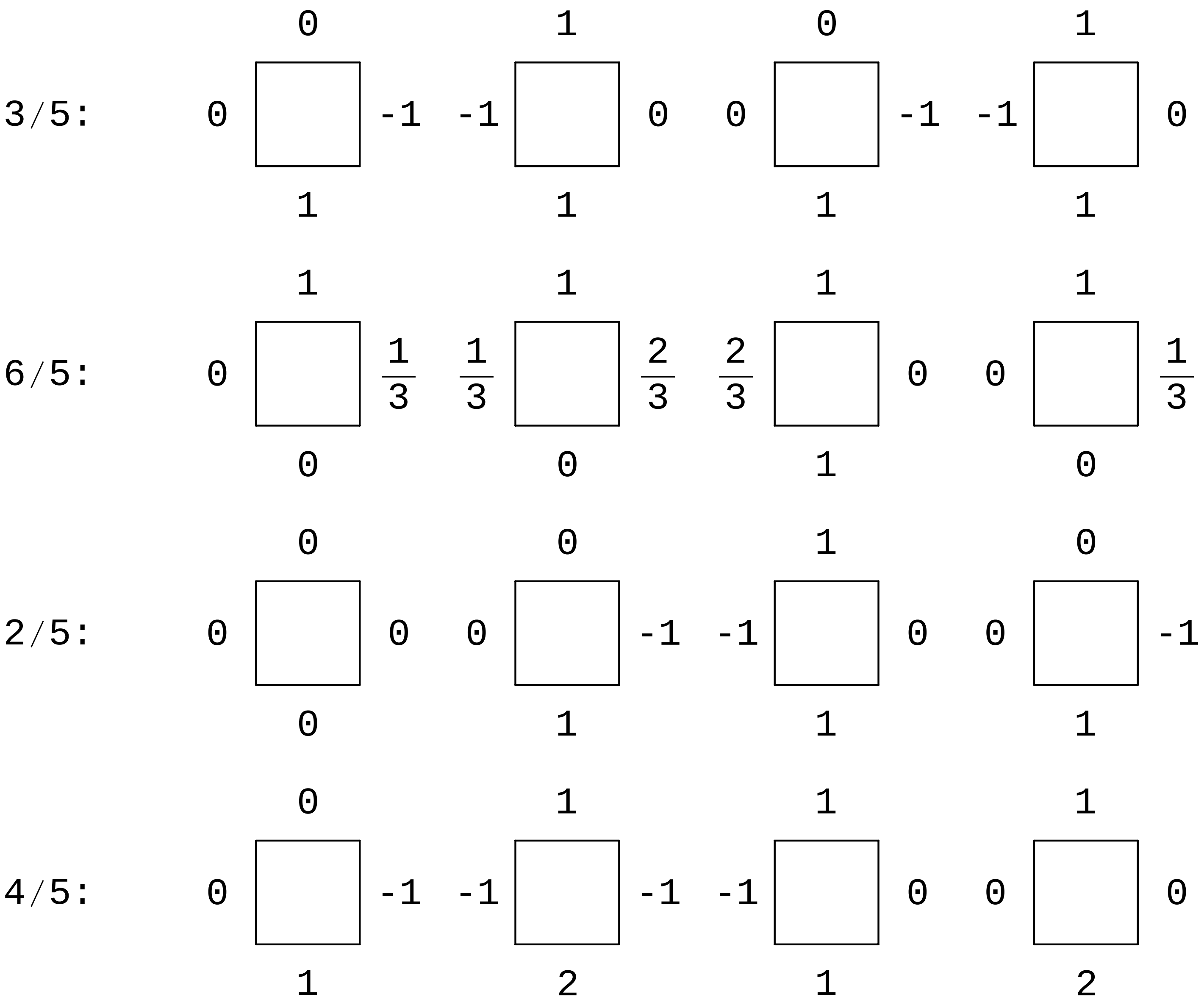


. . . B(q,x,n−1) B(q,x,n) B(q,x,n+1) . . .

- The Top is the Beatty Sequence for x.
- The Bottom is Beatty Sequence for qx.

Illustration: $\frac{3}{5} \rightarrow \frac{6}{5} \rightarrow \frac{2}{5} \rightarrow \frac{4}{5}$,

Beatty tiles $n = 1, 2, 3, 4$



- Beatty Tiles "fit" together to tile the plane.

•• Row for $f(x)$ fits under row for x.

Notation $f(x) = q \cdot x$ (and $f(f(x)) = q_2 \cdot f(x)$)

...	B(q,x,n−1)	B(q,x,n)	B(q,x,n+1)	...
...	B(q_2,f(x),n−1)	B(q_2,f(x),n)	B(q_2,f(x),n+1)	...

Beatty Difference Sequences

The **Beatty Difference Sequence** $B(x)$ **of** x is the two-sided sequence

$$\{\lfloor nx\rfloor - \lfloor (n-1)x\rfloor, \quad n \in Z\}$$

Fact: $x \in [k, k+1)$ then $B(x) \in \prod_{-\infty}^{\infty}\{k, k+1\}$.

Beatty difference sequences and Beatty sequences $\{\lfloor nx\rfloor\}$ $n \in Z$ (named for S. Beatty [1]), are related to continued fraction expansions.

The general Beatty sequence is $\{\lfloor nx+y\rfloor\}$, $n \in Z$.

References to Beatty and Beatty Difference Sequence occur in dynamical systems, the work of J. Bernoulli and go all the way back to the Eudoxan theory of proportions. There are connections to Penrose tilings, formal language theory and computer vision. In the literature they are also referred to as Sturmian sequences and characteristic sequences. Rather than give a large list of references we give only one [4] from which further references can be attained.

Theorem 4 *Let g be any piecewise, rationally multiplicitive, invertible function. Let $\mathcal{T}_g$ be the set of tiles constructed for g. Then every two-sided infinite orbit of g corresponds to a valid tiling of the plane using the tile set $\mathcal{T}_g$.*

Tweaking the Colors of tiles

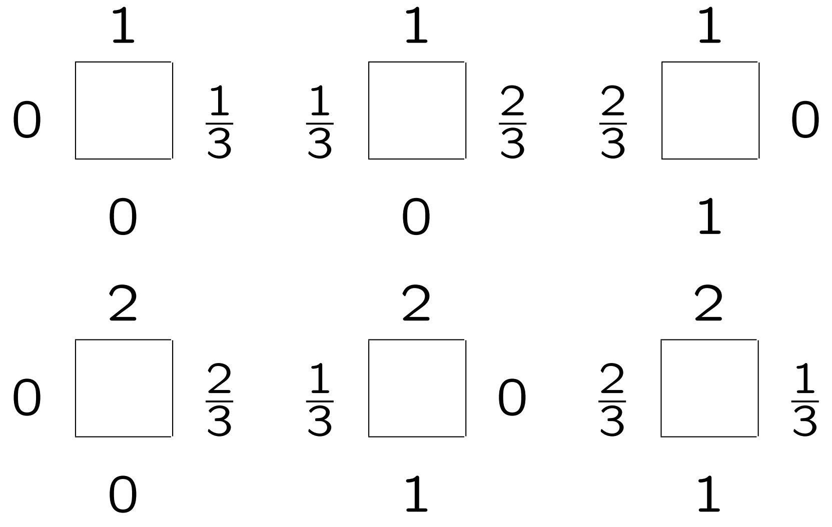
Issue: If we construct the tiles as above we do not get the Aperodicity result.

Hence, the Colors Need to be Tweaked.

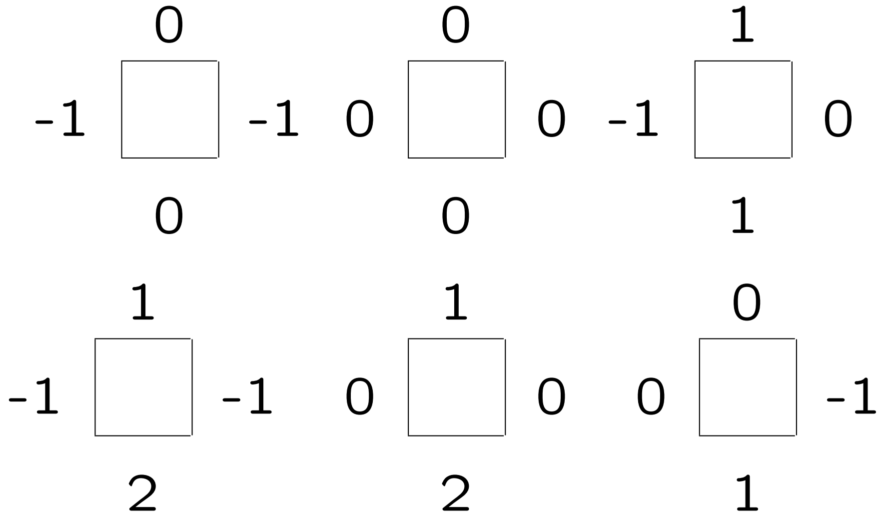
Example A has two color changes.

That is, there are the three "zeros" $\{\frac{0}{3}, 0', 0\}$

Beatty Tiles for $f(x) - NoTweaking$



$*\frac{1}{3}$ Multiplier Tiles

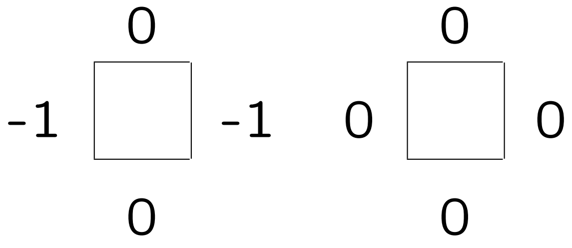


$*2$ Multiplier Tiles

Examing the tiles we observe

1: 0 appears as a Side Color for both the $*\frac{1}{3}$ and $*2$.

2: There are two tiles in $*2$ which can each tile periodically



Side Color Change $0 \rightarrow \frac{0}{3}$

The purpose of changing the color 0 to color $\frac{0}{3}$ is to ensure that each row corresponds to a single multiplicand.

Top-Bottom Color Change

The color change $0'$ from 0 in the tiles for Example A is a top-bottom color change.

The Issue is that the set of tiles $*2$ is not actually the tiles for $2x \;\; x \in [\frac{1}{3}, 1)$.

It is the tile set for $2x \;\; x \in [0, 1)$.

And, this has a fixed point $x = 0$.

Observe:

The dynamical system $f(x)$ can have at most two consecutive iterates in $[\frac{1}{3}, 1)$.

There can be at most two consecutive multiplications by 2.

Tiles of Example A can have at most two consecutive rows from Tile Set $2'$.

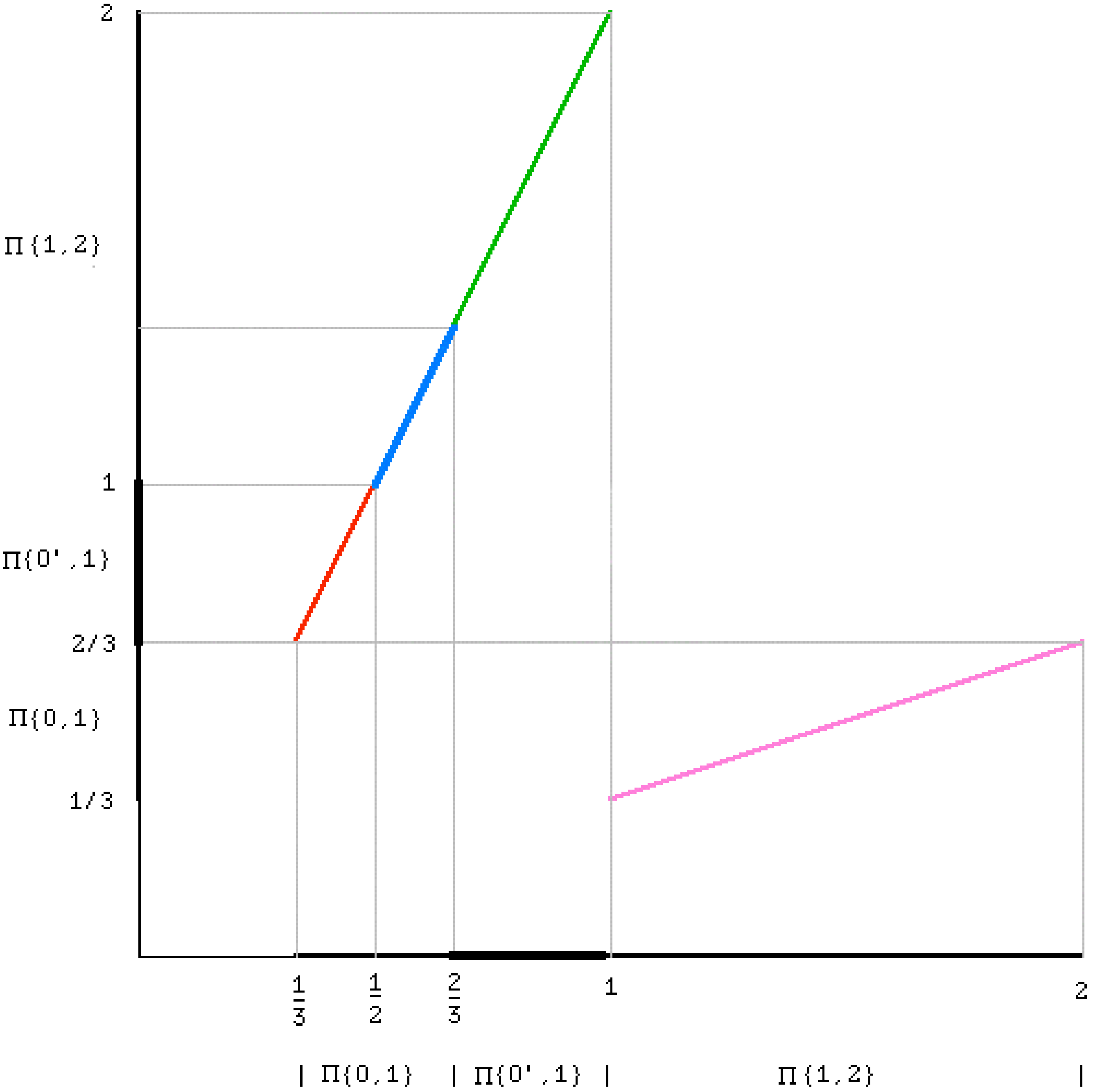
That is, the tile set has been "tweaked" to reflect the "consecutiveness" of the dynamical system.

Rewrite the function f in four pieces as

$$F(x) = \begin{cases} F_1(x) = 2x, & [\frac{1}{3}, \frac{1}{2}) \to [\frac{2}{3}, 1) \\ \\ F_2(x) = 2x, & [\frac{1}{2}, \frac{2}{3}) \to [1, \frac{4}{3}) \\ \\ F_3(x) = 2x, & [\frac{2}{3}, 1) \to [\frac{4}{3}, 2) \\ \\ F_4(x) = \frac{1}{3}x, & [1, 2) \to [\frac{1}{3}, \frac{2}{3}) \end{cases}$$

Observe that it is only piece F_1 that has points with $\{x, F_1(x)\} \in [0, 1)$. Consequently it is only this piece that gives rise to tiles with 0 on both the top and bottom.

In this case, we will make only one color change and that is on the interval $[\frac{2}{3}, 1)$ which is the range of F_1.


2
Π{1,2}
1
Π{0',1}
2/3
Π{0,1}
1/3
$\frac{1}{3}$
$\frac{1}{2}$
$\frac{2}{3}$
1
2
| Π{0,1} | Π{0',1} | Π{1,2} |

Specifically, any $x \in [\frac{2}{3}, 1)$ has a Beatty Difference sequence using just $0, 1$. We change this 0 to $0'$. That is, for any point $x \in [\frac{2}{3}, 1)$, $\lfloor nx \rfloor - \lfloor (n-1)x \rfloor \in \{0', 1\}$.

$$\begin{cases} F_1 : & \prod\{0,1\} \to \prod\{0',1\} \\ F_2 : & \prod\{0,1\} \to \prod\{1,2\} \\ F_3 : & \prod\{0',1\} \to \prod\{1,2\} \\ F_4 : & \prod\{0,1\} \to \prod\{0,1\} \end{cases}$$

This Changes tops of Beatty tiles: $x \in [\frac{2}{3}, 1)$

And Changes bottoms of tiles: $f(x) \in [\frac{2}{3}, 1)$.

Theorem 5 *Let g be a piecewise, rationally multiplicitive, invertible function such that*

$q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k} = 1$ for $n_i \geq 0$ only if $n_i = 0$ for all $i = 1, \cdots, k$;

there is an $M \geq 0$ such that the longest consecutive orbit wholly contained in $[0, 1)$ is of length M.

Then by incorporating both side and top-bottom color changes the resulting tile set, $\mathcal{T}_g$, is aperiodic.

Questions About Wang Tiles

Does a given set have a Valid Tiling? (Are there classes of tile sets for which an algorithm exists.)

If a valid tiling exist is there a Periodic Tiling? (Are there classes of tiles sets for which this can be answered.)

What is the smallest number of tiles in a set which has only Aperiodic valid tilings?

What is the smallest number of colors in a set which has only Aperiodic valid tilings.

Can other types of dynamical systems be used to create small sets of aperiodic tiles? (The suggestion here is to pursue the connections of K. Schmidt regarding Wang Tilings and Cocylces.)

What can be obtained from the Prime-Composite example (mentioned earlier in an aside)?

Give a proof of Robinson's lost theorem

Is the tile set in Example A **minimal**? That is can a subset of it still tile the plane?

Can the tiles in Example A give a tiling which does not come from an orbit of a point? If so, does this give information about the tiling?

*

References

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