Indag. Mathem., N.S., 14 (1), 11–14 March 31, 2003

Solution to a problem of Sands on the factorization of groups

by S.J. Eigen and V.S. Prasad

Northeastern University, Boston, MA 02115
e-mail: eigen@neu.edu
University of Massachusetts Lowell, Lowell, MA 01854
e-mail: vidhu_prasad@uml.edu

Communicated by Prof. M.S. Keane at the meeting of February 24, 2003

ABSTRACT

Using elementary methods, a positive answer is given to a question of A. D. Sands concerning factorizations of abelian groups. We then indicate how our approach to Sands's question has its roots in a result on the ergodic theory of infinite measure preserving transformations due to Eigen, Hajian and Ito.

1. SANDS'S PROBLEM AND AN ELEMENTARY SOLUTION

In [2], A.D. Sands notes that a positive answer to the following problem would simplify the proofs in [2]. Our Theorem 1.1 which is proved using elementary arguments, provides (as a special case) such a positive answer. Here is Sands's problem:

Problem(Sands). *Suppose that $G = A + B$ is a factorization of the group G, and that the subset A is finite. Suppose that a subset C of G exists such that $|A| = |C|$ and the sum of C and B is direct. Does it follow that $G = C + B$?*

The terminology from [2] is as follows: the "group" G is an additive abelian group; the cardinality of a set C is denoted $|C|$; for subsets A, B of G their *sum* is $A + B = \{a + b : a \in A, b \in B\}$. When each element of $A + B$ is expressed uniquely in this way, the sum is called *direct* and we write $A + B = A \oplus B$. A *factorization* of G, means that for some sets A and B, $G = A \oplus B$.

In this note, we write $-A = \{-a : a \in A\}$ and $A - A = A + (-A)$, the *differ-*

ence set of A. When $A + B = G$ (not necessarily direct) we say the sum is *exhaustive.*

The following result contains a positive answer to Sands's question.

Theorem 1.1. *Let $G = A \oplus B$ be a factorization of the abelian group G with $|A| < \infty$. Consider the three conditions on a subset $C \subset G$.*

i) $|C| = |A|$.

ii) $C + B = C \oplus B$; *i.e., the sum is direct.*

iii) $C + B = G$; *i.e., the sum is exhaustive.*

Then, whenever a subset C satisfies any two of the above conditions it must satisfy the remaining condition.

If $|A| = \infty$ the conclusion to Sands's question need not hold (just let C be A with one element removed, for example).

If $|G| < \infty$ the result is trivial. The implication that conditions ii) and iii) imply condition i) does not require that $|A| < \infty$.

The theorem can actually be derived from a similar result in [1] on the ergodic theory of infinite measure preserving transformations. However, we first give a complete and elementary proof without any reference to ergodic theory.

The theorem is an immediate consequence of the following lemma on elementary properties for sums. Except for (8) we do not require $|A| < \infty$.

Lemma 1.2. *Let A, B, C be subsets of the abelian group G.*

(0) $A + B = A \oplus B$ *if and only if* $(A - A) \cap (B - B) = \{0\}$.

(1) *If* $A + B$ *is direct then so is* $A + (-B)$.

(2) *If* $A + B$ *is direct, then* $(A + g) + B$ *is direct for each* $g \in G$.

(3) *If* $A + B = G$ *then* $(A + g) + B = G$ *for all* $g \in G$.

(4) *If* $A + B = A \oplus B$ *then* $|C| \geq \sum_{b \in B} |C \cap (A + b)|$.

(5) *If* $A + B = G$ *then* $|C| \leq \sum_{b \in B} |C \cap (A + b)|$.

(6) *If* $A \oplus B = G$ *then* $|C| = \sum_{b \in B} |C \cap (A + b)|$.

(7) *If* $A + B = A \oplus B$ *and* $C + B = G$ *then* $|A| \leq |C|$.

(8) *If* $A \oplus B = G$ *and* $|A| < \infty$ *then* $A \oplus (-B) = G$.

Proof. (0), (1), (2), (3), (4), (5) and (6) are clear from the definitions. To prove (7), we use

$$\begin{aligned} |A| = |A \cap G| &= |A \cap (\cup_{b \in B}(C + b))| \\ &\leq \sum_{b \in B} |A \cap (C + b)| \\ &= \sum_{b \in B} |(A - b) \cap C| \\ &\leq |C|. \end{aligned}$$

(8) was proved by Sands (Theorem 2, [2]) by extending (Theorem 1, [2]) a technique of Tijdeman [3]. Here we derive the result directly. To prove (8) we need only show the $A + (-B)$ is exhaustive, as (2) implies that the sum is direct. For each $g \in G$ we have

$$\begin{aligned} |A| = |A \cap G| &= |A \cap \cup_{b \in B}((A+g)+b)| \\ &= |\cup_{b \in B}((A-b) \cap (A+g))| \\ &= \sum_{b \in B} |(A-b) \cap (A+g)| \\ &\leq |A+g| = |A| < \infty. \end{aligned}$$

Hence the inequality must be an equality, and so the disjoint sum $\sum_{b \in B} |(A - b) \cap (A+g)|$ counts every member in $A+g$. From this we conclude that $A + g \subset \cup_{b \in B}(A-b)$ (disjoint) $= A \oplus (-B)$. Since this holds for all $g \in G$ and $\cup_{g \in G}(A+g) = G$ we have $A \oplus (-B) = G$.

Proof *(of Theorem 1.1)*. There are three implications to prove, i) and ii) $\Rightarrow$ iii) (this answers the Sands question), i) and iii) $\Rightarrow$ ii), and ii) and iii) $\Rightarrow$ i). The first two implications require that $|A| < \infty$.

First assume $|A| < \infty$, and using (2),(3),(8) we will need the following fact (for the first two implications) which is true for each $g \in G$

$$\begin{aligned} |C| = |C \cap G| &= |\cup_{b \in B}((A+g-b) \cap C)| \\ &= \sum_{b \in B} |(A+g-b) \cap (C)| \\ &= \sum_{b \in B} |(A+g) \cap (C+b)|. \end{aligned}$$

i) and ii) $\Rightarrow$ iii): Assuming that $|C| = |A| < \infty$ and $C + B = C \oplus B$, then for each $g \in G$

$$\begin{aligned} |C| = |C \cap G| &= \sum_{b \in B} |(A+g) \cap (C+b)| \\ &= |\cup_{b \in B}(A+g) \cap (C+b)| \\ &\leq |A+g| = |A| < \infty. \end{aligned}$$

This says $A + g \subset C + B$ for each $g \in G$ and since $\cup_{g \in G}(A+g) = G$, it follows that $C + B = G$, *i.e.* iii). This answers Sands's question.

i) and iii) $\Rightarrow$ ii): Now we assume $|C| = |A| < \infty$, and $C + B = G$.

$$\begin{aligned} \infty > |C| = |C \cap G| &= \sum_{b \in B} |(A+g) \cap (C+b)| \\ &\geq |(A+g) \cap (\cup_{b \in B}(C+b))| \\ &= |A+g| = |A| \end{aligned}$$

This means that the intersection of $C + b$ and $C + b'$ for $b \neq b'$ must be empty on $A + g$ for every $g \in G$. Since $\cup_{g \in G} A + g = G$, then $(C+b) \cap (C+b') = \emptyset$ if $b \neq b'$, i.e., (ii).

ii) and iii) $\Rightarrow$ i): This implication is an immediate consequence of two applications of (7). First, $A+B = A \oplus B$ and $C+B = G$ imply $|A| \leq |C|$. The reverse inequality $|C| \leq |A|$ also follows from (7), by simply interchanging the roles of A and C. Note that we do not require that $|A| < \infty$ for this last implication in the Theorem.

2. ERGODIC THEORY

Our approach to Sands's questions has its roots in the ergodic theory of infinite measure preserving transformations. Let (X, μ) be an infinite measure space. Let T be an invertible measure preserving transformation of (X, μ), $T : (X, \mu) \to (X, \mu)$. A subset $W \subset X$ is said to be *weakly wandering* for a sequence B of the integers if the collection of sets $\{T^b(W) : b \in B\}$ is pairwise disjoint. The set W is called *exhaustive* for the sequence B if $X = \cup_{b \in B} T^b(W)$. The following theorem on exhaustive weakly wandering sets was proved by Eigen, Hajian and Ito [1].

Theorem 2.1 (Eigen–Hajian–Ito [1]). *Let T be an ergodic infinite measure preserving transformation of the sigma finite measure space (X, μ). Let $W \subset X$ be an exhaustive and weakly wandering set for the sequence of integers B. Suppose $\mu(W) < \infty$. Consider the three conditions for some subset $V \subset X$*

i) $\mu(V) = \mu(W)$.

ii) *V is weakly wandering for B.*

iii) *V is exhaustive for the set B.*

Then, when a subset V satisfies any two of the above conditions, it must satisfy the third. Furthermore, the implication that ii*) and* iii*) imply* i*) does not require the assumption that $\mu(W) < \infty$.*

We have adapted the Eigen-Hajian-Ito proof to prove Theorem 1.1 (and therefore answers Sands's question) when $X = G$, μ is counting measure on the set of all subsets of G, and $T_g(x) = x + g$ for $x \in G$, $W = A$ and $V = C$. Note that strictly speaking, the Theorem above does not apply to prove Theorem 1.1 since the G-action is not ergodic.

REFERENCES

[1] Eigen, S., A. Hajian and Y. Ito – Ergodic measure preserving transformations of finite type. Tokyo J. Math. **11**, no. 2, 459–470 (1988).

[2] Sands, A.D. – Replacement of factors by subgroups in the factorization of abelian groups. Bull. London Math. Soc. **32**, 297–304 (2000).

[3] Tijdeman, R. – Decomposition of the integers as a direct sum of two subsets. Number theory, (ed. S. David), Cambridge University Press, 261–276 (1995).