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Universally bad integers and the 2-adics

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Abstract

In his 1964 paper, de Bruijn (Math. Comp. 18 (1964) 537) called a pair (a, b) of positive odd integers good, if $\mathbb{Z} = a\mathbb{S} \ominus 2b\mathbb{S}$, where $\mathbb{S}$ is the set of nonnegative integers whose 4-adic expansion has only 0's and 1's, otherwise he called the pair (a, b) bad. Using the 2-adic integers we obtain a characterization of all bad pairs. A positive odd integer u is universally bad if (ua, b) is bad for all pairs of positive odd integers a and b. De Bruijn showed that all positive integers of the form $u = 2^k + 1$ are universally bad. We apply our characterization of bad pairs to give another proof of this result of de Bruijn, and to show that all integers of the form $u = \phi_{p^k}(4)$ are universally bad, where p is prime and $\phi_n(x)$ is the nth cyclotomic polynomial. We consider a new class of integers we call de Bruijn universally bad integers and obtain a characterization of such positive integers. We apply this characterization to show that the universally bad integers $u = \phi_{p^k}(4)$ are in fact de Bruijn universally bad for all primes $p > 2$. Furthermore, we show that the universally bad integers $\phi_{2^k}(4)$, and more generally, those of the form $4^k + 1$, are not de Bruijn universally bad.

Keywords: Cyclotomic polynomials; 2-adic integers; Bases for integers; Tiling the integers; Universally bad integers

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1. Introduction

Consider the set $\mathbb{S} = \{0, 1, 4, 5, 16, 17, 20, 21, \ldots\}$ constructed by taking sums of finite subsets of the even powers of 2 (0 corresponding to the sum over the empty subset). It is well known that the integers $\mathbb{Z}$ can be written as $\mathbb{Z} = \mathbb{S} \ominus 2\mathbb{S}$ (which means that each integer n can be written as a difference, $n = s_1 - 2s_2$ with $s_1, s_2 \in \mathbb{S}$, in exactly one way).

In connection with problems on bases for the integers, de Bruijn used this relation between $\mathbb{S}$ and $\mathbb{Z}$, to define what he called in [3], good pairs of positive integers (nota bene: what de Bruijn called a good pair in [3], he previously called a basic pair in [2]).

Definition 1. A pair of positive, odd integers (a, b) is *good* if $\mathbb{Z} = a\mathbb{S} \ominus 2b\mathbb{S}$; otherwise, the pair is *bad*.

Following de Bruijn, we note that the pair (a, b) is good if and only if the formal relation

$$F(x^a)F(x^{-2b}) = \sum_{k=-\infty}^{\infty} x^k$$

holds where $F(x)$ is the following infinite product:

$$F(x) = \prod_{k=0}^{\infty}(1 + x^{2^{2k}}).$$

Some examples of good pairs include (1,1), (1,7), and (7,13). A list of some good pairs (a, b) can be found in [3] (this 1964 list was made with the aid of a computer, and it extends his earlier 1950 list in [2] for $1 \leqslant a \leqslant b \leqslant 100$ obtained by "pencil and paper" and "shuffling four strips of paper"—these quotes are from de Bruijn's 1964 paper [3]; see our acknowledgement at the end of our paper). De Bruijn gave some conditions for a pair to be good: For example, he observed [2] that $\gcd(a, b) = 1$ is a necessary condition; he showed (a, b) is good if and only if (b, a) is good; in [3], de Bruijn gives a procedure which, for each pair (a, b), determines in a finite number of steps whether the pair is good or not.

We use the 2-adic integers to give a complete characterization of all bad (and therefore all good) pairs in our Theorem 12. This characterization follows from an elementary consideration in Section 2 of the 2-adic integers $\mathbb{Z}_2$, ending with our basic result (Theorem 10) that for any pair of positive odd integers (a, b), the 2-adic integers can be written as $\mathbb{Z}_2 = \overline{a\mathbb{S} \ominus 2b\mathbb{S}} = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}$ (here the closures are taken in the 2-adic topology).

De Bruijn proved [2, Theorem 8], that any integer $u = 2^k + 1, k \geqslant 1$ satisfies the following: for each pair of positive odd integers (a, b), the pair (ua, b) is bad. We call such integers u, universally bad.

Definition 2. A positive integer u is *universally bad* if the pair (ua, b) is bad for all positive odd integers a and b.

Note that when u is even, it is trivially universally bad: u is even implies the difference set $ua\mathbb{S} - 2b\mathbb{S}$ contains only even integers for positive odd pairs (a, b). It is only among the positive odd integers that we find nontrivial universally bad integers. We note that if u is universally bad, then for any (odd) x, $(u(xa), b)$ is bad for all positive odd pairs (a, b); thus any multiple of a universally bad integer is also universally bad.

We apply our characterization of bad pairs to give a new proof of de Bruijn's $2^k + 1$ result. The argument also shows:

Theorem 3. *All integers $u = \phi_{p^k}(4)$ are universally bad when p is prime, $k \geqslant 1$, and $\phi_n(x)$ is the n'th cyclotomic polynomial.*

Our approach leads us to consider a new class of universally bad integers which we call de Bruijn universally bad integers (see Section 3). Theorem 19 characterizes the de Bruijn universally bad integers and we use this characterization to prove that the integers $\phi_{p^k}(4)$ are de Bruijn universally bad for any prime p except $p = 2$. In Section 4, we show that the universally bad integers $\phi_{2^k}(4)$, and $4^k + 1$ are not de Bruijn universally bad. We end the paper with some questions for further research.

The reader should note that we use only elementary properties of the 2-adic integers and cyclotomic polynomials. We provide, in Section 2, the basic properties of the 2-adic integers; in Section 3, we summarize the necessary properties concerning cyclotomic polynomials (see also [5]).

2. The 2-adic integers

Our analysis uses the 2-adic integers whose properties we recall in this section (see [1] or [8] for details).

Definition 4. For a nonnegative integer $n \geqslant 0$ the 2-adic *order* of n is defined by $ord(n) = k$, if k is the highest power of 2 which divides n. The 2-adic *norm* of $n \geqslant 0$ is defined as $|n| = 2^{-ord(n)}$. We follow convention by setting $ord(0) = \infty$ and $|0| = 0$.

We denote the 2-adic integers by $\mathbb{Z}_2 = \{z = \sum_{i \geqslant 0} z_i 2^i : z_i \in \{0, 1\}\}$, the completion of the nonnegative integers (denoted by $\mathbb{N}$) in the 2-adic norm. We denote by $\overline{\mathbb{S}}$, the closure of $\mathbb{S}$ in this norm. For notational convenience we identify $\mathbb{Z}_2$ with $\{0, 1\}^{\mathbb{N}}$, using the correspondence $z = \sum z_i 2^i \leftrightarrow (z_0 z_1 z_2 \cdots)$; z_i is the entry in the ith coordinate (or the ith place) of z—note that the coordinate numbering starts at 0. We extend $ord : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ from the integers, to the 2-adic integers, by setting $ord(z) = k$, the coordinate of the first nonzero entry z_k of $z = \sum z_i 2^i = (z_0 z_1 \cdots) \in \mathbb{Z}_2$. Define the 2-adic norm of $z \in \mathbb{Z}_2$, by setting $|z| = 2^{-ord(z)}$. This relation between $|z|$ and $ord(z)$ permits the analysis of convergence in terms of $ord(\cdot)$ instead of $|\cdot|$. Thus, a sequence of terms $z(n) \in \mathbb{Z}_2$ converges if and only if $ord(z(n) - z(m)) \to \infty$ as $n, m \to \infty$.

Under this correspondence, the positive integers are represented in the 2-adic integers as $n = (n_0 n_1 n_2 \cdots)$ with $n_i = 0$ for all but finitely many i's; the negative integers are represented by $m = (m_0 m_1 m_2 \cdots)$ with $m_i = 1$ for all but finitely many i's; numbers in $\mathbb{S}$ and its closure $\overline{\mathbb{S}}$ have 0's in all odd places: i.e., $\sigma = (\sigma_0 \sigma_1 \cdots) \in \overline{\mathbb{S}}$ if and only if $\sigma_{2i+1} = 0$ for all $i \in \mathbb{N}$. Addition in this representation is coordinatewise from left to right with "carry" to the right.

The geometric series formula for the 2-adic integers is a valuable tool in our analysis. A rational $z = p/q$, $\gcd(p,q) = 1$, is in $\mathbb{Z}_2$ if and only if q is odd, and in that case, z has an eventually repeating representation in the 2-adic integers given by $z = (z_0 \cdots z_{k-1} \overline{z_k \cdots z_{k+r-1}})$ where the notation $\overline{z_k \cdots z_{k+r-1}}$ indicates this r-block repeats forever.

Lemma 5 (Geometric series). *If $|x| < 1$, (i.e. $ord(x) > 1$) then $\sum_0^\infty x^i = \frac{1}{1-x}$ in the 2-adic integers.*

Here are a few illustrations: The integer -1 has the representation $-1 = (\bar{1}) = \sum_0^\infty 2^i = \frac{1}{1-2}$; the fraction $-\frac{1}{3}$ has the repeating pattern $(\overline{10}) = \sum_0^\infty 4^i = \frac{1}{1-4}$, and in general, $\sum_{i=0}^\infty 4^{in} = \frac{1}{1-4^n}$. Note that these latter two are 2-adic integers in the closure of $\mathbb{S}$.

To distinguish the set $\mathbb{S}$ and its difference set $\mathbb{S} - \mathbb{S} = \{s - s' \mid s, s' \in \mathbb{S}\}$ from other 2-adic integers we consider the following definition.

Definition 6. A 2-adic integer $z \in \mathbb{Z}_2$ is of *even order* if $ord(z) = 2i$ and of *odd order* if $ord(z) = 2i + 1$.

We note that all odd integers a, (positive and negative) are of even order: in fact, $ord(a) = 0$ since the highest power of 2 which divides an odd integer a is 2^0. By convention, 0 is considered both even order and odd order, and is the only such number. The idea of using even order to study complementing pairs can be found in [7,9].

The proofs of the next three lemmas are easy and are left for the reader.

Lemma 7

(a) *s is of even order for all $s \in \mathbb{S}$.*
(b) *$s - s'$ is of even order for all $s, s' \in \mathbb{S}$.*

Furthermore, the lemma extends from $\mathbb{S}$ to $a\mathbb{S}$ where a is a positive odd integer, and to the closure $\overline{\mathbb{S}}$ in the 2-adic norm.

Lemma 8. *Let a be a positive odd integer. Then,*

(c) *$a\sigma$ is of even order for all $\sigma \in \overline{\mathbb{S}}$ and $ord(a\sigma) = ord(\sigma)$.*
(d) *$a\sigma - a\sigma'$ is of even order for all $\sigma, \sigma' \in \overline{\mathbb{S}}$ and $ord(a\sigma - a\sigma') = ord(\sigma - \sigma')$.*

An immediate consequence of these two lemmas is

Lemma 9. *Let $\tau, \tau' \in \overline{2\mathbb{S}}$ and let b be an odd integer. Then,*

(e) *$ord(b\tau) = ord(\tau)$ and each element in $\overline{2\mathbb{S}}$ is of odd order.*
(f) *$ord(b\tau - b\tau') = ord(\tau - \tau')$ and each element in $\overline{2\mathbb{S}} - \overline{2\mathbb{S}}$ is of odd order.*

This odd/even dichotomy between the sets $\mathbb{S}$ and $2\mathbb{S}$ implies the next result which is fundamental to our analysis (see also [6]).

Theorem 10. *For all pairs (a, b) of positive odd integers, the 2-adic integers can be written as*

$$\mathbb{Z}_2 = \overline{a\mathbb{S} \ominus 2b\mathbb{S}} = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}.$$

The theorem will follow from the following three facts which we now prove:

(1) $\overline{a\mathbb{S}} - \overline{2b\mathbb{S}} = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}$: the difference is direct.
(2) $\overline{a\mathbb{S} - 2b\mathbb{S}} = a\overline{\mathbb{S}} - 2b\overline{\mathbb{S}}$: closure distributes over difference.
(3) $\mathbb{Z}_2 = \overline{a\mathbb{S} - 2b\mathbb{S}}$: the closure includes all of the 2-adic integers.

Proof (1) $\overline{a\mathbb{S}} - \overline{2b\mathbb{S}} = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}$.
We show that each difference is unique. Suppose $a\sigma - 2b\tau = a\sigma' - 2b\tau'$. Rearranging the terms gives $a(\sigma - \sigma') = b(2\tau - 2\tau')$. The left-hand side has even order by Lemma 8(d) and the right-hand side has odd order by Lemma 9(f) and so both sides must be 0.

(2) $\overline{a\mathbb{S} - 2b\mathbb{S}} = a\overline{\mathbb{S}} - 2b\overline{\mathbb{S}}$.
One containment, $\overline{a\mathbb{S} - 2b\mathbb{S}} \supset a\overline{\mathbb{S}} - 2b\overline{\mathbb{S}}$, is obvious. The other containment follows from: *Suppose $as(n) - 2bt(n)$ converges in $\mathbb{Z}_2$, with sequences $s(n), t(n) \in \mathbb{S}$ (a, b are positive odd integers). Then $s(n)$ and $t(n)$ each converge.* The latter follows by rewriting the Cauchy difference $as(n) - 2bt(n) - (as(m) - 2bt(m)) = [as(n) - as(m)] - b[2(t(n)) - 2(t(m))]$ and using the fact that when $ord(x) \neq ord(y)$ then $ord(x \pm y) = min(ord(x), ord(y))$. Apply this to $x = [as(n) - as(m)]$ which has even order and $y = b[2(t(n)) - 2(t(m))]$ which has odd order.

(3) $\mathbb{Z}_2 = \overline{a\mathbb{S} - 2b\mathbb{S}}$.
It is enough to show that $a\mathbb{S} - 2b\mathbb{S}$ is dense in $\mathbb{Z}$ which is in turn dense in $\mathbb{Z}_2$.
Fix N and consider the 2^{2N} integers $k \in \{0, 1, \ldots, 2^{2N} - 1\}$, and denote the ball of radius 2^{-2N} around each k, by $B_{k,2N} = \{z : ord(k - z) \geqslant 2N\} = \{z : |k - z| \leqslant 2^{-2N}\}$. The 2^{2N} sets, $\{B_{k,2N} : k = 0, 1, \ldots, 2^{2N} - 1\}$, are disjoint clopen sets and form a partition of $\mathbb{Z}_2$. In particular, every integer (in fact every $z \in \mathbb{Z}_2$) is in one of the clopen sets and must agree with exactly one of the integers $k \in \{0, 1, \ldots, 2^{2N} - 1\}$ in the first $2N$ coordinates.
Each integer k is a unique difference $k = s_k - 2t_k$ with $s_k, t_k \in \mathbb{S}$. Using this, define a map f by setting $f(k) = as_k - 2bt_k$.

It will turn out, from the next lemma, that every integer (and again every $z \in \mathbb{Z}_2$) must agree in the first $2N$ coordinates with exactly one of the 2^{2N} images $f(k)$, $0 \leqslant k < 2^{2N}$. The required lemma follows by rearranging terms and using the odd and even order properties of differences from $\mathbb{S}$.

Lemma 11. *Let $s, t, s', t' \in \mathbb{S}$. Let a, b be positive odd integers. Then $ord((s-2t)-(s'-2t')) = ord((as-2bt)-(as'-2bt'))$.*

Returning to the proof of (3), each image $f(k)$ must be in one of the clopen sets of the partition $\{B_{i,2N}: i = 0, 1, \ldots, 2^{2N}-1\}$. The 2-adic distance between any two distinct integers $k, k' \in \{0, 1, \ldots, 2^{2N}-1\}$ is greater than 2^{-2N}; that is, $ord(k-k') < 2N$. By Lemma 11, their images $f(k), f(k')$ have a distance greater than 2^{-2N} and so must be in different sets in the partition. This completes the proof of Theorem 10. □

3. De Bruijn universally bad integers

Using Theorem 10 we have

Theorem 12. *The pair of positive odd integers (a, b) is bad, i.e. $\mathbb{Z} \neq a\mathbb{S} \ominus 2b\mathbb{S}$, if and only if there is an integer n such that $n = a\sigma - 2b\tau$, where σ or $\tau \in \overline{\mathbb{S}} \backslash \mathbb{S}$.*

Proof. If some integer $n = a\sigma - 2b\tau$, where σ or $\tau \in \overline{\mathbb{S}} \backslash \mathbb{S}$, then by the uniqueness property of the decomposition $\mathbb{Z}_2 = \overline{a\mathbb{S}} \ominus \overline{2b\mathbb{S}}$, there cannot be $s, t \in \mathbb{S}$ with $n = as - 2bt$. Conversely, if the pair (a, b) is bad then there is some integer n which is not in $a\mathbb{S} - 2b\mathbb{S}$. But then Theorem 10 implies that this $n = a\sigma - 2b\tau$ and at least one of σ or τ must be in $\overline{\mathbb{S}} \backslash \mathbb{S}$. □

The next corollary, which is a special case of de Bruijn's result that $2^k + 1$ is universally bad [2, Theorem 8], illustrates how we use Theorem 12.

Corollary 13. *The integer 3 is universally bad.*

Proof. Let (a, b) be any pair of odd positive integers; we need to show that $(3a, b)$ is bad. The fraction $-\frac{1}{3} = (\overline{10})$ belongs to $\overline{\mathbb{S}} \backslash \mathbb{S}$. Putting $\sigma = -\frac{1}{3}$ and $\tau = 0$ we have $-a = 3a\sigma - 2b \cdot 0$ and so by Theorem 12, $(3a, b)$ is bad. □

Consider the following class of integers.

Definition 14. An odd positive integer u is a *de Bruijn universally bad integer* if there is some $\sigma \in \overline{\mathbb{S}} \backslash \mathbb{S}$ such that $u\sigma \in \mathbb{Z}$.

Using Theorem 12 in the same way that it was used in the previous corollary we have:

Corollary 15. *De Bruijn universally bad integers are universally bad.*

From the definition of a de Bruijn universally bad integer, it is obvious that any odd multiple of a de Bruijn universally bad integer is de Bruijn universally bad. Since 3 divides any integer of the form $2^{2k+1}+1$ the following is immediate.

Corollary 16. *For all* $k \geqslant 0$, $2^{2k+1}+1$ *is de Bruijn universally bad.*

Other de Bruijn universally bad integers are given in the next two examples.

Example 17. The integer $u = 85$ is de Bruijn universally bad. To see this, observe that the fraction $-21/255 = (\overline{10101000}) = \frac{1+4+4^2}{1-4^4} \in \overline{\mathbb{S}} \backslash \mathbb{S}$, and $85 \cdot \frac{-21}{255} = -7$.

Example 18. The integer $u = 341$ is de Bruijn universally bad. In this case, the fraction $-\frac{81}{1023} = (\overline{1000101000}) = \frac{1+4^2+4^3}{1-4^5}$ is in $\overline{\mathbb{S}} \backslash \mathbb{S}$ and $341 \cdot \frac{-81}{1023} = -27$.

The integer 341 is a new universally bad integer not on de Bruijn's list [3]. It is a product of 11 and 31 neither of which is universally bad. The integer 85 is on de Bruijn's list since it is a multiple of $5 = 2^2+1$ and $17 = 2^4+1$, which are universally bad, although neither 5 nor 17 is a de Bruijn universally bad integer—we show the latter fact in the next section.

These examples are special cases of the following:

Theorem 19. *A positive odd integer u is a de Bruijn universally bad integer if and only if there exists a fraction of the form* $\sigma = \frac{\sum_{i=0}^{R-1} \delta_i 4^i}{1-4^R} \in \overline{\mathbb{S}} \backslash \mathbb{S}$ *with* $\delta_i \in \{0,1\}$, *and such that* $u\sigma \in \mathbb{Z}$.

Proof. If u is a de Bruijn universally bad integer, and $u\alpha = n$ with $\alpha \in \overline{\mathbb{S}}$, then $\alpha = n/u$ and so is "rational". It remains to show the there is a σ of the form above; i.e. possessing a purely repeating 2-adic expansion, such that $u\sigma \in \mathbb{Z}$.

The 2-adic representation of $\alpha = n/u$ is $(\alpha_0\alpha_1 \cdots \alpha_{L-1}\overline{\alpha_L\alpha_{L+1}\cdots\alpha_{L+R-1}})$. Since $\alpha \in \overline{\mathbb{S}}$, then $\alpha_{2i+1} = 0$ for all i, and α is the infinite sum

$$\alpha = \sum_{l=0}^{L-1} \varepsilon_l 4^l + \left(4^L \sum_{i=0}^{R-1} \delta_i 4^i\right)\left(\sum_{j=0}^{\infty} 4^{jR}\right),$$

where $\varepsilon_l, \delta_i \in \{0,1\}$ and $\delta_0 = 1$.

Multiply by u to get the integer n

$$u\alpha = u\sum_{i=0}^{L-1} \varepsilon_l 4^l + u\left(4^L \sum_{i=0}^{R-1} \delta_i 4^i\right)\left(\sum_{j=0}^{\infty} 4^{jR}\right) = n.$$

A simple subtraction yields

$$u\left(4^L \sum_{i=0}^{R-1} \delta_i 4^i\right)\left(\sum_{j=0}^{\infty} 4^{jR}\right) = n - u\sum_{l=0}^{L-1} \varepsilon_l 4^l.$$

The right-hand side is an integer, which we claim is divisible by 4^L. Indeed, the left-hand side is a product of 4 terms: u which is a positive odd integer; 4^L which is a power of 2; $\sum_0^{R-1}\delta_i 4^i$ which is a positive odd integer since $\delta_0 = 1$; $\sum_0^{\infty} 4^{jR} = \frac{1}{1-4^R}$ which is a fraction with an odd integer (negative) in the denominator. Since the product of the 4 terms is an integer, the product must be divisible by 4^L.

Dividing by 4^L we have $\sigma = \left(\sum_0^{R-1} \delta_i 4^i)/(1-4^R\right)$ is as required since

$$u\left(\sum_0^{R-1} \delta_i 4^i\right)\left(\sum_0^{\infty} 4^{jR}\right) = \left(n - u\sum_0^{L-1} \varepsilon_l 4^l\right)\Big/ 4^L,$$

which is still an integer.

The converse is trivial. □

We extend this argument to prove that the positive integers $\phi_{p^k}(4)$ are de Bruijn universally bad when $\phi_n(x)$ is the nth cyclotomic polynomial. Briefly, for $n\geqslant 1$, the nth cyclotomic polynomial is defined recursively by the relation

$$x^n - 1 = \Pi_{\{d\,:\,d|n\}}\phi_d(x),$$

where the factors d of n in the product above, include both 1 and n.

Theorem 20. *The integers $\phi_{p^k}(4)$ are de Bruijn universally bad integers for primes $p>2$, integers $k\geqslant 1$, and $\phi_n(x)$ the nth cyclotomic polynomial.*

Before proceeding with the proof of the theorem, we collect some facts about cyclotomic polynomials (see, for example [5]): for p a prime and integers $k\geqslant 1$,

$$\phi_p(x) = 1 + x + \cdots + x^{p-1}$$

$$\phi_{p^k}(x) = \phi_p(x^{p^{k-1}}) = \sum_{i=0}^{p-1} x^{ip^{k-1}}.$$

The proof of the theorem also uses the following fact which follows immediately from the expression above for $\phi_{p^k}(x)$. For p prime, and $k \geqslant 1$,

$$\phi_{p^k}(x)\phi_{p^{k-1}}(x)\cdots\phi_p(x) = \sum_0^{p^k-1} x^i.$$

As a consequence of Theorem 20, we have that $\phi_5(4) = 341$ (which was shown earlier to be a new universally bad integer) is a de Bruijn universally bad integer.

We will show in the next section that $\phi_{2^k}(4)$ is universally bad but is not de Bruijn universally bad.

Proof of Theorem 20. To begin with, observe that $\phi_{3^k}(4) = 1 + 4^{3^{k-1}} + 4^{2\cdot 3^{k-1}}$ is always divisible by 3, and so is a de Bruijn universally bad integer. We therefore assume that p is a prime and $p > 3$.

Then,

$$\sigma = \frac{(1 + 4^{p^{k-1}} + 4^{2p^{k-1}})\sum_0^{p^{k-1}-1} 4^i}{1 - 4^{p^k}} = \frac{\sum_0^{3p^{k-1}-1} 4^i}{1 - 4^{p^k}} \in \overline{\mathbb{S}} \backslash \mathbb{S}.$$

So,

$$\begin{aligned}
\phi_{p^k}(4)\sigma &= \phi_{p^k}(4)\frac{\left(1 + 4^{p^{k-1}} + 4^{2p^{k-1}}\right)\sum_0^{p^{k-1}-1} 4^i}{1 - 4^{p^k}} \\
&= \left(1 + 4^{p^{k-1}} + 4^{2p^{k-1}}\right)\frac{\phi_{p^k}(4)\sum_0^{p^{k-1}-1} 4^i}{1 - 4^{p^k}} \\
&= \left(1 + 4^{p^{k-1}} + 4^{2p^{k-1}}\right)\frac{\sum_0^{p^k-1} 4^i}{1 - 4^{p^k}} \\
&= \frac{1 + 4^{p^{k-1}} + 4^{2p^{k-1}}}{1 - 4} \\
&\in \mathbb{Z}
\end{aligned}$$

because $1 + 4^{p^{k-1}} + 4^{2p^{k-1}}$ is divisible by 3. $\square$

Using properties of the cyclotomic polynomials we can get additional de Bruijn universally bad integers. First, we introduce some notation. For a finite set of nonnegative integers $\mathbb{A} = \{0 = a_1 < a_2 < \cdots < a_n\}$ define the polynomial $A(x) = \sum_1^n x^{a_i}$.

Corollary 21. *Let* $\mathbb{A} = \{0 = a_1 < a_2 < \cdots < a_n\}$ *be a finite set of integers whose cardinality* n *is divisible by some prime* $p > 2$. *If there exists another finite set of integers* $\mathbb{B} = \{0 = b_1 < b_2 < \cdots < b_m\}$ *such that* $\mathbb{A} \oplus \mathbb{B} = \{0, 1, \ldots, nm - 1\}$. *Then* $A(4)$ *is de Bruijn universally bad.*

Proof. Under these hypotheses, de Bruijn's work [4] on tiling the non-negative integers implies there must be a $k \geqslant 1$ so that $\phi_{p^k}(x)|A(x)$ (see also the related result in Coven–Meyerowitz [5, Lemma 1.3]). Thus $A(4)$ is a multiple of the de Bruijn universally bad integer $\phi_{p^k}(4)$. □

4. More universally bad integers

De Bruijn showed that each integer in the sequence $2^k + 1$ is universally bad, and we have just seen that half of these, namely those of the form $2^{2k+1} + 1$ for $k \geqslant 0$ are de Bruijn universally bad integers. In this section we show that the remaining integers (i.e., those of the form $2^{2k} + 1$, for $k \geqslant 1$) are universally bad while not being de Bruijn universally bad. We begin by presenting the proof for $u = 5$. This contains all the essential ideas.

Proposition 22. *The integer* $5 = 2^2 + 1$ *is not de Bruijn universally bad.*

Proof. If $5\alpha = n \in \mathbb{Z}$ for some $\alpha \in \mathbb{Z}_2$, then $\alpha = n/5$. We show that any number of the form $n/5$, where $5 \nmid n$, is not in $\overline{\mathbb{S}} \backslash \mathbb{S}$. Our method, is to look at the 2-adic representation of $\alpha = (\alpha_0 \alpha_1 \alpha_2 \cdots)$, and use the fact that any number in $\overline{\mathbb{S}}$ cannot have a 1 in any odd coordinate (we remind the reader that in our numbering, α_0 is the entry in the 0th coordinate).

First $-1/5$ is not in $\overline{\mathbb{S}}$, since the 2-adic representation of the fraction $-\frac{1}{5} = (\overline{1100})$ has 1 in coordinate 1. The 2-adic representations of the fractions $-\frac{2}{5} = (0\overline{1100})$, $-\frac{3}{5} = (100\overline{1100})$, $-\frac{4}{5} = (00\overline{1100})$ all have 1 in an odd coordinate and so are not in $\overline{\mathbb{S}}$.

Now, adding any positive integer m to one of these four fractions gives $m - \frac{i}{5}$ (for $1 \leqslant i \leqslant 4$) whose 2-adic representation follows. Since m is a positive integer its 2-adic representation must end in all 0's.

$$\begin{array}{r l}
m = & (m_0 m_1 \cdots 00000000 \cdots) \\
-i/5 = & (f_0 f_1 \cdots \quad 11001100 \cdots) \\
\hline
m - i/5 = & (\quad \cdots \quad 11001100 \cdots)
\end{array}$$

Looking at the sum $m - \frac{i}{5}$ and recalling that addition is coordinatewise from left to right with carry to the right, we see (because there are two 0's in a row in the 2-adic expansion of $-i/5$) that there can only be a finite number of carries in the addition above, and hence the sum $m - \frac{i}{5}$ must end in the repeated block $\overline{1100}$. Hence, all numbers $m - \frac{i}{5}$ for $m > 0$ and $i = 1, 2, 3, 4$ are not in $\overline{\mathbb{S}}$.

A similar argument holds if $m < 0$. In this case, m ends in all 1's and therefore from some coordinate on, when m is added to $-i/5$, there is always a carry to the right and as above, the sum $m - i/5$ ends in the repeated block $\overline{1100}$. □

The next lemma is required for both the proposition and theorem which follow it.

Lemma 23. *For any odd positive integer b which is not a multiple of* 3 *there is a number in* $-2b(\overline{\mathbb{S}}\backslash\mathbb{S})$ *with a fractional part either* $\frac{1}{3}$ *or* $\frac{2}{3}$.

Proof. Since $\frac{2}{3}\in -2(\overline{\mathbb{S}}\backslash\mathbb{S})$, then $\frac{2b}{3}\in -2b(\overline{\mathbb{S}}\backslash\mathbb{S})$ is our required element since b is not a multiple of 3. □

Proposition 24. *The integer* 5 *is universally bad.*

Proof. Let (a,b) a pair of positive odd integers. We show $(5a,b)$ is bad. Without loss of generality, we can assume that neither a nor b is divisible by 3, otherwise Corollary 13 and the fact that (u,v) is bad if and only if (v,u) is bad, would imply that $(5a,b)$ would be a bad pair. The two numbers $-\frac{1}{3}$ and $-\frac{1}{15}=(1000\overline{1000})=\sum 4^{2i}$ are in $\overline{\mathbb{S}}\backslash\mathbb{S}$. Multiplying by 5 we see that $-\frac{5}{3}$ and $-\frac{1}{3}$ are both in $5(\overline{\mathbb{S}}\backslash\mathbb{S})$. Therefore, since a is not a multiple of 3, the set of fractional parts of the two numbers $-\frac{a}{3}$ and $-\frac{5a}{3}$ is $\{1/3, 2/3\}$. The previous lemma shows that, for $\tau=-\frac{1}{3}\in(\overline{\mathbb{S}}\backslash\mathbb{S})$, the fractional part of $-2b\tau$ is $1/3$ or $2/3$. Choose $\sigma\in 5a(\overline{\mathbb{S}}\backslash\mathbb{S})$ to be either $-\frac{a}{3}$ or $-\frac{5a}{3}$, so that $\sigma-2b\tau\in\mathbb{Z}$. Using Theorem 12, $(5a,b)$ is bad. □

Theorem 25. *For all* $k\geqslant 1$, *the odd integers* $u=2^{2k}+1=4^k+1$ *are universally bad but not de Bruijn universally bad.*

Proof. First to show that 4^k+1 is universally bad (using Lemma 23 in the same way it was used in the previous proposition), it is enough to show that $(1+4^k)\cdot(\overline{\mathbb{S}}\backslash\mathbb{S})$ contains two numbers one with fractional part $-\frac{1}{3}$ and another with fractional part $-\frac{2}{3}$.

Observe that $1-4^{2k}=(1-4^k)(1+4^k)=(1-4)(\sum_0^{k-1}4^i)(1+4^k)$. Put $\sigma=\frac{\sum_0^{k-1}4^i}{1-4^{2k}}\in\overline{\mathbb{S}}\backslash\mathbb{S}$. Consider $(1+4^k)$ times the two numbers $\sigma, -\frac{1}{3}\in\overline{\mathbb{S}}\backslash\mathbb{S}$. The first product is $(1+4^k)\sigma=\frac{(1+4^k)\sum_0^{k-1}4^i}{1-4^{2k}}=-\frac{1}{3}$. The second product is $(1+4^k)(-\frac{1}{3})=-(\frac{4^k-1}{3}+\frac{2}{3})=-\left(\sum_0^{k-1}4^i+\frac{2}{3}\right)$. Hence, $u=4^k+1$ is universally bad.

Finally, we show that 4^k+1 is not de Bruijn universally bad; that is $(1+4^k)\cdot\sigma\notin\mathbb{Z}$ for every $\sigma\in\overline{\mathbb{S}}\backslash\mathbb{S}$. This follows if we show that the 2-adic representation of $(1+4^k)\sigma=(z_0z_1\cdots)$ cannot end in all 0's or all 1's.

If $\sigma=(\sigma_0\sigma_1\cdots)\in\overline{\mathbb{S}}\backslash\mathbb{S}$, then for infinitely many i's, $\sigma_{2i}=1$ and for all i, $\sigma_{2i+1}=0$. In $(1+4^k)\sigma$ the even coordinates of σ and $4^k\sigma$ are being added (with a possible carry to the right). Therefore, infinitely often, either two 1's add giving a 0 in an even place and a 1 in the next odd place, or a 1 and a 0 add giving a 1 in an even place and a 0 in the next place. Either way, $\sigma+4^k\sigma$ cannot end in all 0's or all 1's, and is thus not in $\mathbb{Z}$.

$$\begin{array}{r l}
\sigma = & (\sigma_0 \cdots \sigma_{4^k} \cdots 10 \cdots 10 \cdots) \\
4^k\sigma = & (\qquad \sigma_0 \cdots 10 \cdots 00 \cdots) \\
\hline
(1+4^k)\sigma = & (\quad \cdots \qquad \cdots 01 \cdots 10 \cdots)
\end{array}$$

This completes the proof of this theorem. □

Finally, noting that $\phi_{2^k}(4) = \phi_2(4^{2^{k-1}}) = 1 + 4^{2^{k-1}}$, we have

Corollary 26. *The integers $\phi_{2^k}(4)$ are universally bad but not de Bruijn universally bad.*

5. Questions about universally bad integers

We end this note with some questions for further investigation, and an acknowledgement of our admiration of de Bruijn's work.

Question 1. *If u is a de Bruijn universally bad integer, is it divisible by 3 or $\phi_{p^k}(4)$ for some p and k?*

Question 2. *The integers $4^k + 1$ (and their multiples) are universally bad but not de Bruijn universally bad. What other integers are universally bad but not de Bruijn universally bad?*

Question 3. *What is the (upper and lower) density of the universally bad integers? Assuming the density exists, it is certainly more than 1/3.*

Question 4. *Given an odd positive integer a which has at least one b with which it is a good pair, are there infinitely many b's with which it is a good pair? De Bruijn [3] shows that this is true for $a = 1$.*

Question 5. *Is there an algorithm—meaning some program that stops in a finite number of steps—which determines if a given integer is universally bad?*

Remark. We end these questions with an acknowledgement of our admiration of the work of N.G. de Bruijn (and his four strips of paper). In his 1964 paper [3] de Bruijn gives a procedure, which for a pair of integers (a, b) determines in a finite number of steps if it is a good pair or not. This is done by constructing an oriented graph whose vertices are the integers[1] and then showing that (a, b) is good if and only if the graph is a tree. Referring to his earlier 1950 paper, de Bruijn notes in [3].

[1] If x, x_1 are integers, de Bruijn takes an oriented edge from x to x_1 if and only if one of the following relations hold:

$$x = 4x_1,\ \ x = 4x_1 + a,\ \ x = 4x_1 - 2b,\ \ x = 4x_1 + a - 2b.$$

Removing the loop from 0 to 0, yields the graph referred to above.

"In [2] we listed all good pairs as far as $1 \leqslant a \leqslant b \leqslant 100$, obtained with the aid of pencil and paper. (This included making a table of the relation between x and x_1, constructed with four strips of paper that simply had to be shifted in order to switch on the next pair.)"

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