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Examples and properties of nonexact ergodic shift measures

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ABSTRACT

We construct examples of nonexact n-to-one shifts. We first construct examples of one-sided shift measures with maximal automorphic factors of any prescribed finite cardinality. Then we give an example of a two-to-one shift with a maximal automorphic factor which is an odometer.

1. INTRODUCTION

In this paper we construct ergodic measure preserving n-to-one shifts with various prescribed exactness properties. In particular we are interested in determining which automorphic factors can occur as the maximal automorphic factor of a full n-shift.

J. Feldman conjectured that ergodic nonexact shifts on n states exist, and in this paper we give a general construction which yields many nonisomorphic ergodic nonexact shifts, including ones with uncountable maximal automorphic factors.

Our construction contrasts with earlier studies on the existence of invariant measures for endomorphisms, where the conditions giving the existence also imply that the maximal automorphic factor is trivial or a finite rotation (see for example [11] and [13]). Adler and Weiss [1] showed that the transformation induced by the Boole transformation on $[-1, 1]$ is ergodic but not exact; further studies on exactness properties of the modified Boole transformation were done in [4].

We show that only zero entropy automorphisms can arise as n-shift maximal automorphisms. The question of whether every zero entropy automorphism can occur is still open, but the construction given here can be extended to include many zero entropy automorphic factors of shifts; some extensions arising from this construction have been shown to the authors by K. Petersen.

The outline of the paper is as follows. We begin by describing two factors that are present in any nonsingular endomorphism. The first is called a Rohlin factor; it is trivial if and only if the endomorphism T is invertible with respect to the given measure class, and a full n-shift when T is n-to-one. The second factor is the maximal automorphic factor; this factor is unique, and is trivial if and only if the endomorphism is exact.

In Section 2 we describe a general construction of an ergodic measure μ for a one-sided shift on n states which is not exact. The construction can be modified to give bounded-to-one endomorphisms as well as n-to-one. Its maximal automorphic factor is a rotation on k states, for any prescribed $k \in \mathbb{N}$. We also show that for any ergodic nonsingular n-shift, the maximal automorphic factor has zero entropy. In Section 3 we construct an example of a two-to-one shift whose maximal automorphic factor is the dyadic odometer, and discuss generalizations and open questions.

1.1. Nonsingular n-to-one maps

Throughout this paper we will assume that all spaces $(X, \mathfrak{B}, \mu)$ are Lebesgue probability spaces and that all maps T are forward and backward measurable and nonsingular; that is, for all $A \in \mathfrak{B}$, TA, $T^{-1}A \in \mathfrak{B}$, and $\mu(T^{-1}A) = 0 \Longleftrightarrow \mu(A) = 0 \Longleftrightarrow \mu(TA) = 0$. We say that a measurable map $T : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{B}, \mu)$ is an n-to-one nonsingular endomorphism if there exists a partition $\mathcal{P}$ of X into exactly n atoms of positive measure, $P_0, \dots, P_{n-1}$, each atom P_i having the property that the restriction of T to P_i is one-to-one, nonsingular, and onto X. (This is also called *essentially n-to-one* by Walters, [19]); equivalently, T is n-to-one if $\mu - a.e.$ point x has exactly n preimages under T [14].

Example 1. We define the space $X = X_n^+ = \prod_{i=0}^{\infty} \{0, \dots, n-1\}_i$ and we put the usual Borel structure on X. Let T denote the one-sided shift on X. We consider any nonatomic Borel measure μ on X with respect to which T is n-to-one, ergodic, and nonsingular. It then follows that $\mu(C) > 0$ for C any cylinder of finite length on X (cf. [2]). In this case we call T a full n-shift.

Not every n-to-one ergodic (or even exact) transformation is isomorphic to a full n-shift (cf. [2]).

The n-shift example is ubiquitous in the sense that every n-to-one endomorphism contains a full n-shift as a factor, called a Rohlin factor.

1.2. Rohlin partitions and Rohlin factors

If T is n-to-one, any partition $\mathcal{P} = \{P_0, P_1, \dots, P_{n-1}\}$ with the property that the restriction T_i of T to P_i is one-to-one, nonsingular, and onto X is called a *Rohlin* *partition* for T. If ϵ denotes the point partition of X, then $\mathcal{P}$ separates points in each atom of $T^{-1}\epsilon$; i.e., $\mathcal{P} \vee T^{-1}\epsilon = \epsilon$. A Rohlin partition is not unique, although Parry [12] gives a canonical method of choosing it. Let $\mathcal{P}$ be a Rohlin partition; by $J_\mu T(x)$ we denote the Jacobian function at x (as defined by Parry [12]), so $(d\mu T_i/d\mu)(x) = J_\mu T(x)$ for each $x \in P_i$. Parry has shown that the value of $J_\mu T(x)$ is independent of the partition $\mathcal{P}$.

A *Rohlin factor* is the factor generated by a Rohlin partition. It is a factor which carries significant information about the noninvertibility and entropy of the original endomorphism. If we denote a Rohlin factor by $(Z, \mathfrak{F}, \rho)$, then ρ is the restriction of μ to $\mathfrak{F} \subseteq \mathfrak{B}$, the smallest sub-$\sigma$-algebra generated by $\vee_{i=0}^{\infty} T^{-i}\mathcal{P}$.

In Example 1, T is isomorphic to its Rohlin factor using $P_i = \{x : x_0 = i\}$, while in general a Rohlin factor is strictly contained in the original system, and is not known to be unique. In what follows we will construct nonexact ergodic Rohlin factors.

1.3. The maximal automorphic factor and orbit relations

We turn to the definition of the maximal automorphic factor of a nonsingular endomorphism.

Given a nonsingular endomorphism T on $(X, \mathfrak{B}, \mu)$, we define the *tail field* of T by $\cap_{i \geq 0} T^{-i}\mathfrak{B}$. Clearly $\mathfrak{B} = \cap_{i \geq 0} T^{-i}\mathfrak{B}$ (up to sets of μ measure 0) if and only if T is invertible. The factor map of T on $(X, \cap_{i \geq 0} T^{-i}\mathfrak{B}, \mu)$ is defined to be the *maximal automorphic factor* of T. Since a natural projection exists from T on X to its maximal automorphic factor, we sometimes use the notation $(Y, \mathfrak{D}, \nu)$ to denote the image of X under the projection, and call the factor map S.

In our construction we use the relationship between the invertible odometer map (addition by 1 on the left with a carry – also called the adding machine and the von Neumann transformation), and the shift map T on the same one-sided product space, X_n^+, for each $n > 1$. If μ is an ergodic nonsingular measure for the odometer, then μ is nonsingular for the shift if and only if the shift is nonsingular and exact for the measure μ [7], [14]. When μ is nonsingular for both maps then the shift is exact if and only if the odometer is ergodic [7]. Thus constructing a nonergodic odometer is equivalent to constructing a shift with a nontrivial maximal automorphic factor. The difficulty is in maintaining invariance (or even nonsingularity) of the measure for the shift in the construction.

This connection between an invertible and a noninvertible map on X_n^+ is made precise by the following orbit relations for an endomorphism. We refer to [7] or [5] for details. We use these relations in our construction.

Assume that T is a nonsingular endomorphism of $(X, \mathfrak{B}, \mu)$. We define $R_T = \{(x, w) \in X \times X : T^n x = T^m w \text{ for some } m, n \in \mathbb{N}\}$; this is often called the *grand orbit relation* of T.

Similarly, we define the subrelation $S_T = \{(x, w) \in X \times X : T^n x = T^n w \text{ for some } n \in \mathbb{N}\}$. These relations are measurable amenable relations for countable-to-one T, and using the notion of a (measurable) quotient relation defined by Feldman, Sutherland, and Zimmer [5], it can be shown that the ergodic decomposition of the grand orbit relation by S_T gives, in a natural way, the group $\mathbb{Z}$ acting on the tail field of T by the maximal (invertible) automorphic factor. The group $\mathbb{Z}$, endowed with the action, is called the *quotient relation* R_T/S_T [3].

2. THE GENERAL CONSTRUCTION

In this section, we present a general construction of a measure for the one-sided shift on n states with control over the maximal automorphic factor. The construction is motivated by one given by Dajani and Hawkins [2].

2.1. Notation and outline of the construction

We begin with two spaces with maps denoted $(X; T)$ and $(Y; R)$. On $X \times Y$ we construct a $T \times R$ invariant measure μ. This measure is then projected onto X to define an invariant measure σ for T. With minimal assumptions, σ will be ergodic, n-to-1 for T, and in our later examples we will be able to control its maximal automorphic factor. To be more precise, this factor will inherit all its properties from $(Y; R)$.

The measure μ on $X \times Y$ is constructed fiberwise. Specifically, we put a measure λ on Y and fiber measures ρ_y on each fiber $X \times \{y\}$. The choice of fiber measures ρ_y is measurable in y; i.e., for any Borel set $B \in \mathfrak{B}$, the map $y \mapsto \rho_y(B)$ is a measurable function on Y. Then for each Borel product set $B \times F$ we put

$$\mu(B \times F) = \int_F \rho_y(B) d\lambda(y).$$

This measure is extended in the usual way to the product Borel σ-algebra denoted $\mathfrak{B} \times \mathfrak{F}$. By the usual abuse of language, for any $E \in \mathfrak{B} \times \mathfrak{F}$ we write

$$\mu(E) = \int_Y \rho_y(E) d\lambda(y).$$

We project the measure μ onto X, and in a later section we will prove that the choices can be made so that the projection is, after removing a set of measure zero, injective.

It remains to describe explicitly the construction of the fiber measures ρ_y. Let $n > 1$ and $k > 1$ be fixed integers. Throughout this paper X will denote the product space $X = X_n^+ = \prod_{i=0}^{\infty} \{0, 1, \ldots, n-1\}_i$ with the shift T.

We consider any ergodic process $(Y; R, \mathcal{Q})$; i.e., a Lebesgue probability space $(Y, \mathfrak{F}, \lambda)$, an invertible ergodic measure preserving transformation R, and a finite generating partition $\mathcal{Q} = \{Q_0, \ldots, Q_{k-1}\}$ with k atoms.

We set $\mathbb{N}_n = \{0, 1, \ldots, n-1\}$, and we choose k probability n-tuples $\beta_0, \cdots, \beta_{k-1}$ on $\mathbb{N}_n$. Thus $\beta_i = \{\beta_i^{(0)}, \ldots, \beta_i^{(n-1)}\}$ where $\beta_i^{(j)} \geq 0$ and $\sum_{j=0}^{n-1} \beta_i^{(j)} = 1$ for all $i = 0, \ldots, k-1$.

We associate to each point $y \in Y$ its one-sided $\mathcal{Q}$-name $y \mapsto (y_0 y_1 y_2 \ldots)$

where $y_i = j$ if $R^i(y) \in Q_j$. Using its Q-name, we associate to each y a product measure ρ_y on the fiber $X \times \{y\}$ by

$$\rho_y = \prod_{i=0}^{\infty} \beta_{y_i}.$$

Now we define the measure μ on $X \times Y$ fiberwise as described above, for all $E \in \mathfrak{B} \times \mathfrak{F}$.

Finally, we project the measure μ onto X, with factor measure σ, and the resulting factor map on X will be denoted (T, σ).

2.2. Properties of the general construction

We now present some fundamental results about the construction above which follow from the definitions or are in [2].

Lemma 2. *The measure μ on $X \times Y$ projects to λ on Y.*

Lemma 3. *The map $T \times R$ takes the fiber $X \times \{y\}$ onto the fiber $X \times \{Ry\}$ and $\rho_{Ry} \circ T^{-1} = \rho_y$.*

Lemma 4. *The measure μ is invariant for $T \times R$.*

Lemma 5. *The factor measure σ on X is invariant for T.*

Lemma 6. *If every β_i has no zero component; i.e., $\beta_i^{(j)} \neq 0$ for all i, j, then $T \times R$ is forward and backward nonsingular with respect to μ on $X \times Y$. If some $\beta_i^{(j)} = 0$, then there exists a set $Z \subset X \times Y$ such that $\mu(Z) = 1$, Z is $T \times R$ invariant, and $T \times R$ is forward and backward nonsingular on (Z, μ).*

Proof. The backward nonsingularity for all β_i is obvious. For the forward nonsingularity, for each $y \in Y$ we remove a set of ρ_y measure 0 from each fiber in a measurable way. Specifically, we replace $X \times \{y\}$ by $\zeta_y = \prod_{i=0}^{\infty} X_i^{(y_i)}$, where $j \in X_i^{(y_i)} \subset \mathbb{N}_n$ if and only if $\beta_{y_i}^{(j)} \neq 0$. We then define $Z = \prod_{y \in Y} \zeta_y$. It is clear that Z is measurable and has full measure in $X \times Y$, and $Z = X \times Y$ precisely when all β_i have no zero component. The nonsingularity now follows since for every $(x, y) \in Z$, $J_\mu(T \times R)(x, y) = 1/\beta_{y_0}^{(x_0)} \neq 0$. □

From now on we assume that $T \times R$ is both forward and backward nonsingular (so that we have removed a set of measure zero from $X \times Y$ if necessary). For simplicity of notation, we will still refer to the space as $X \times Y$ unless confusion arises. We will also always mean forward and backward nonsingular when we say nonsingular.

Recalling from the introduction the relations $S_{T \times R}$ and S_T, for any measurable set $A \in X \times Y$, we denote $S_{T \times R}(A) = \{(w, z) : ((x, y), (w, z)) \in S_{T \times R}$ for some $(x, y) \in A\}$, and we say that $S_{T \times R}$ is nonsingular with respect to μ if $\mu(A) = 0 \Leftrightarrow \mu(S_T(A)) = 0$.

The importance of the nonsingularity of $T \times R$ is apparent from the following lemma [7].

Lemma 7. *Under the assumptions of the general construction, the relation $S_{T \times R}$ is nonsingular w.r.t. μ if and only if $T \times R$ is nonsingular w.r.t. μ.*

Proof. ($\Longleftarrow$): This is easy using the forward and backward nonsingularity of $T \times R$. ($\Longrightarrow$): This follows from Lemmas 4 and 5 and the hypothesis. $\square$

We note that the converse of Lemma 7 is false for general endomorphisms. A nonsingular measure for a dyadic odometer, for example, does not typically give a nonsingular measure for a one-sided 2-shift [9]. However if T denotes the full 2-shift, then the relation S_T is exactly the orbit relation of the odometer [7].

Lemma 8. *Under the assumptions of the general construction, $((x, y), (w, z)) \in S_{T \times R}$ if and only if $y = z$ and $(x, w) \in S_T$; i.e., $S_{T \times R} = S_T \times Id_Y$ where Id_Y is just the trivial relation on Y.*

Although T does not give a nonsingular action on an individual fiber $X \times \{y\}$, we nevertheless have the following result. The proof follows immediately from the nonsingularity of $T \times R$.

Lemma 9. *Under the assumptions of the general construction, S_T is nonsingular w.r.t. ρ_y for λ a.e. $y \in Y$.*

Using the notation [3], we call a measure ρ for T *tail trivial* if the tail field of T is $\{\emptyset, X\}$ ρ mod 0; i.e., ρ is tail trivial if and only if every measurable set C satisfying $\rho(C \,\Delta\, T^{-n} \circ T^n C) = 0$ satisfies $\rho(C) = 0$ or 1. Since ρ is not necessarily nonsingular for T, we distinguish it from an exact measure.

Lemma 10. *For λ a.e. $y \in Y$, ρ_y is tail trivial for T.*

Proof. Since each ρ_y is a product measure the Zero One Law can be applied as in [8] to obtain tail triviality. $\square$

Corollary 11. *For λ a.e. $y \in Y$, the following holds: For any measurable set B in X such that $B \in \cap_{i \geq 0} T^{-i}\mathfrak{B}$, either $\rho_y(B) = 0$ or $\rho_y(B) = 1$.*

A related result is the following which is proved in [7].

Lemma 12. *Let X denote the one-sided shift space on n states, and let σ be any nonsingular measure for the odometer on X. Then σ is ergodic for the shift T on X if and only if T is nonsingular and exact with respect to σ.*

It is well-known (cf. [14]) that, up to sets of measure 0, tail sets for a nonsingular endomorphism T are in one-to-one correspondence with S_T invariant sets. In particular the following holds.

Proposition 13. *Under the assumptions of the general construction, the following are equivalent: for some* $A \in \mathfrak{B} \times \mathfrak{F}$;

1. $\mu(S_{T\times R}(A) \,\Delta\, A) = 0$;
2. $A \in \bigcap_{n\geq 0}(T \times R)^{-n}\mathfrak{B} \times \mathfrak{F}$;
3. $\mu(A \,\Delta\, (X \times D)) = 0$ *for some* $D \in \mathfrak{F}$.

Lemma 14. *(R, λ) on Y is the maximal automorphic factor of $(T \times R, \mu)$.*

Proof. Proposition 13 gives the result immediately. □

Corollary 15. *The tail field of (T, σ) is a factor of the tail field of (R, λ). That is, the maximal automorphic factor of T is a factor of R.*

Our hypotheses guarantee ergodicity in our construction.

Theorem 16. *$(T \times R, \mu)$ is ergodic.*

Proof. Suppose that A is an invariant set of positive measure for $T \times R$. Since any invariant set is a tail set, then by Proposition 13 we can change A by a set of measure 0 if necessary so that A is a rectangle in $X \times Y$ of the form $X \times D$, and by ergodicity of R, $\lambda(D) = 1$, hence $\mu(A) = 1$. □

Corollary 17. *(T, σ) is ergodic.*

The above results were for arbitrary β_i; we now examine some specific restrictions on the β_i.

Proposition 18.

1. *If the β_i are trivial (take only the values* 0 *and* 1*) and all distinct, then (T, σ) is a factor of (R, λ). In particular, if R has zero entropy then Q is a one-sided generator and (T, σ) is isomorphic to (R, λ).*
2. *If $k = 1$, then Y is a one-point space and (T, σ) is a one-sided Bernoulli shift.*
3. *If the β_i are all equal, then μ is a direct product measure and (T, σ) is Bernoulli.*
4. *If $\beta_i^{(j)} \neq 0$ for all i,j then $(T \times R, \mu)$ is n-to-*1 *and (T, σ) is n-to-*1.

Proof. 1 follows since each ρ_y is atomic on the point $x_i = y_i$. 2 and 3 are obvious. To prove 4, we set $C_i = \{x : x_o = i\}$, $i = 0, \ldots n-1$. We claim that the sets $P_i = C_i \times Y$ form a Rohlin partition on $X \times Y$. This follows since $J_\mu(T \times R)(x, y) = 1/\beta_{y_o}^{(x_o)} \neq 0$ for μ a.e. $(x, y) \in P_i$. This shows that $T \times R$ is n-to-one. Similarly we show that $C_i = \{x : x_0 = i\}$, $i = 0, \ldots, n-1$ is a Rohlin partition for T. □

Remark. Between the above extremes of n-to-1 and 1-to-1, all the bounded-to-one combinations can occur by choosing some of the β_i to have zero entries.

2.3. The entropy of the maximal automorphic factor

Throughout this section we will assume that $T : (X, \mathfrak{B}, \sigma) \to (X, \mathfrak{B}, \sigma)$ is an endomorphism which is n-to-one, ergodic, and preserves σ. We assume in addition that there exists a Rohlin partition $\mathcal{P}$ which generates $\mathfrak{B}$. Therefore we can assume without loss of generality that T is a one-sided full shift on an n-state space and that $h_\mu T \leq \log n$.

We consider the maximal automorphic factor of T. We will denote it by $(Y, \mathfrak{D}, \nu; S)$; that is, $\mathfrak{D} = \cap_{k \geq 0} T^{-k}\mathfrak{B}(\sigma \bmod 0)$ and S is just the map T restricted to the atoms of $\mathfrak{D}$.

Theorem 19. *Under the assumptions above, $h_\nu(S) = 0$.*

Proof. Since S is a factor of T, it has finite entropy and is invertible, so there exists a finite generating partition call it $\mathcal{A}$, and $h(\mathcal{A}, S) = h_\nu(S)$. Clearly

$$\mathcal{A} \subset \mathfrak{D} = \cap_{k \geq 0} T^{-k}\mathfrak{B}(\text{mod } 0) = \cap_{k \geq 0} T^{-k}\left(\bigvee_{i \geq 0} T^{-i}\mathcal{P}\right) = Tail(\mathcal{P}).$$

It is well-known then that $h(\mathcal{A}, S) = 0$ (cf. [16] or [17]). □

Using this result it is easy to construct examples of $T \times R$ with exact Rohlin factor (T, σ).

Corollary 20. *If T is any n-to-one ergodic measure-preserving endomorphism (not necessarily an n-shift) whose maximal automorphic factor is a K-automorphism, then every Rohlin factor of T is exact.*

Proof. If the Rohlin factor is not exact, then there is an automorphic factor of T which has 0 entropy. But this is impossible if the maximal one is a K-automorphism and hence contains no zero entropy factors. □

We rephrase the corollary in the language of our general construction given above.

Corollary 21. *If (R, λ) on Y is a K-automorphism, and at least one β_i is non-trivial ($0 < \beta_i^{(j)} < 1$ for at least one i,j), then (T, σ) is exact.*

A Bernoulli example. The general construction described above was first used to construct a non-product two-to-one measure for $T \times R$ with an exact Rohlin factor [2]. For that example we choose (R, λ) to be a 2-sided Bernoulli shift, and the β_i to be distinct and have no zero components; then the projection from $X \times Y$ onto X is one-to-one. The important feature of (R, λ) on Y is that R is of positive entropy. This means that the generating partition Q is a two-sided generator. Therefore each one-sided Q name can correspond to more than one point $y \in Y$.

From now on we will assume that the $\beta_i^{(j)} > 0$ and so all the transformations are n-to-1.

2.4. Carriers of measures and examples with finite rotation factors

In order to construct an n-to-one transformation with prescribed automorphic factor, we begin with the general construction outlined above on a product space $X \times Y$. We then remove a set of measure zero from the product space and project the measure μ from $X \times Y$ onto X. The resulting example will then be realized as (T, σ) on X. In this section, we construct the first example of a nonexact ergodic shift. The example has a two-point rotation as a maximal factor. The exposition of this simpler example is meant to clarify the idea used in the uncountable example.

An example of a 2-shift with a maximal automorphic factor equal to a rotation on 2 points

We set $n = 2$, $(Y; R) = (\{0, 1\}; y \to y + 1 \bmod 2)$. The measure we put on Y is the $\{\frac{1}{2}, \frac{1}{2}\}$ measure. We set $Q_o = \{0\}$ and $Q_1 = \{1\}$. We choose two n-tuples $\beta_0 = \{\frac{1}{2}, \frac{1}{2}\}$ and $\beta_1 = \{\frac{1}{3}, \frac{2}{3}\}$, (but any distinct nonzero vectors will work). The measure ρ_0 on $X \times \{0\}$ is the product measure

$$\rho_0 = \beta_0 \times \beta_1 \times \beta_0 \times \ldots = \{\tfrac{1}{2}, \tfrac{1}{2}\} \times \{\tfrac{1}{3}, \tfrac{2}{3}\} \times \{\tfrac{1}{2}, \tfrac{1}{2}\} \times \ldots$$

and ρ_1 on $X \times \{1\}$ is the product measure

$$\rho_1 = \beta_1 \times \beta_0 \times \beta_1 \times \ldots = \{\tfrac{1}{3}, \tfrac{2}{3}\} \times \{\tfrac{1}{2}, \tfrac{1}{2}\} \times \{\tfrac{1}{3}, \tfrac{2}{3}\} \times \ldots$$

These two measures, when viewed on the same space X are easily seen to be mutually singular ([9]).

We now define the measure $\mu = \begin{cases} \frac{1}{2}\rho_0 & \text{on } X \times \{0\} \\ \frac{1}{2}\rho_1 & \text{on } X \times \{1\} \end{cases}.$

To help distinguish the fiber measures on X we introduce the following terminology.

Definition 22. A carrier for the measure μ is any set $C \in \mathfrak{B} \times \mathfrak{F}$ of μ measure one. A carrier for the measure ρ_y, denoted as $C_y \times \{y\}$, $C_y \subset X$, is any set of ρ_y measure one.

(We denote carriers of ρ_y by $C_y \times \{y\}$ to enable us to view the carriers as subsets of the same X.)

Lemma 23. *There exists a* $T \times R$ *invariant carrier* C *for* μ *such that* $C = C_0 \times \{0\} \cup C_1 \times \{1\}$, *and the sets* C_0, C_1 *on* X *are disjoint (i.e., their intersection is the empty set).*

Proof. We define the carrier by

$$C = \{(x,y) \mid \begin{cases} C_0 = \{x \mid \begin{cases} \lim \frac{1}{n+1} \sum_0^n x_{2i} = \frac{1}{2} \\ \lim \frac{1}{n+1} \sum_0^n x_{2i+1} = \frac{2}{3} \end{cases} & y = 0 \\ \\ C_1 = \{x \mid \begin{cases} \lim \frac{1}{n+1} \sum_0^n x_{2i} = \frac{2}{3} \\ \lim \frac{1}{n+1} \sum_0^n x_{2i+1} = \frac{1}{2} \end{cases} & y = 1 \end{cases} \qquad \square$$

By removing a set of measure zero from $X \times Y$ we obtain the following result about the projection of $(X \times Y, \mu)$ onto (X, σ).

Theorem 24. *In the above example the projection* $(X \times Y, \mu) \to (X, \sigma)$ *is* 1-*to*-1; *that is, it is an invertible, measure preserving map.*

Proof. This follows easily from the disjointness of C_1 and C_2 on X. We remove the complement of C, then we apply the projection and consider an image point $x \in X$. Since only one of $(x, 0)$ and $(x, 1)$ is in C, and the other is in C^c, we know exactly which fiber x came from under the projection. This proves the result. $\square$

A similar proof yields the following theorem.

Theorem 25. *For the general construction as defined in Section* 2.1, *if there exist disjoint carriers for the* ρ_y, *then the projection from* $(X \times Y, \mu)$ *to* (X, σ) *is an isomorphism.*

Corollary 26. *For the general construction defined in Section* 2.1, *if there exist disjoint carriers for the* ρ_y, *then the maximal automorphic factor of the shift* T *on* (X, σ) *is* R *on* (Y, λ).

Proof. We have already shown in Lemma 14 that (R, λ) is the maximal automorphic factor of $(T \times R, \mu)$. Theorem 25 shows that (T, σ) is isomorphic to $(T \times R, \mu)$, hence the result follows. $\square$

2.5. Examples of *n*-shifts with *k*-state factors

It is straightforward to generalize the example of the previous section to an ergodic shift on $n > 1$ states with maximal automorphic factor of k-point rotation on $k > 1$ atoms.

Let $(Y;R)$ be the rotation on k points and Q the k-set partition into points. We need only to specify the β_i. We define $\mathbb{Z}_k = \prod_{i=0}^{k-1} \{0,\dots,n-1\}_i$. We fix k ordered n-tuples $\beta_i = \{\beta_i^{(j)}\}, i = 0,\dots,k-1, j = 0,\dots,n-1$. The β_i are viewed as measures on $\mathbb{N}_n = \{0,\dots,n-1\}$. We now define k atomic measures on $\mathbb{Z}_k$ by

$$\eta_p = \prod_{i=0}^{k-1} \beta_{p+i \pmod k}$$

for $p = 0\dots k-1$.

Definition 27. The β_i satisfy Condition β if the k measures η_p are distinct.

For example, $\beta_p \neq \beta_q$ for all $p \neq q$ is sufficient.

The fiber measures, and measure on the product space are defined as before. This results in k measures ρ_p, $p = 0,\dots,k-1$, on X, or more precisely on different fibers of $X \times \{y\}$ as follows:

$$\rho_p = \prod_{i=0}^{\infty} \beta_{p+i \pmod k}$$

Classical results of Kakutani [9] yield the following.

Lemma 28. *If the β_i satisfy Condition β:*

1. *The measures ρ_p and ρ_q are mutually singular whenever $p \neq q$.*
2. *The shift map T takes ρ_p to ρ_{p+1} for $p = 0,\dots k-2$, and ρ_{k-1} to ρ_0.*
3. *The shift T is singular with respect to each measure ρ_p.*

As in the 2-point case, we can find a carrier C for the measure μ such that the projection onto X is 1-to-1 *a.e.* In this case it is easy to see that the measure μ projects to $\sigma = \frac{1}{k}\sum_{p=0}^{k-1} \rho_p$ on X.

Theorem 29. *If the β_i satisfy Condition β then the measure σ is an ergodic invariant measure for the shift T which has the k-point rotation as its maximal automorphic factor.*

Remark 30. 1. In the above examples, the shift is not totally ergodic with respect to μ. In particular, T^k is not ergodic. In general, in order for the Rohlin factor of a countable-to-one map to be exact, the map must be totally ergodic.

2. We will show in a later paper that under additional hypotheses on the Jacobian of σ, this is the only type of example which can occur for ergodic n-to-one shifts. This result uses the Yosida-Kakutani Uniform Ergodic Theorem [20], and is closely related to a result of Rychlik [13].

In this section we construct an example of a two-to-one measure preserving shift with an ergodic measure σ whose maximal automorphic factor is isomorphic to the measure-preserving odometer on a two-state space.

As before we have $X = X_2^+$ and T is the one-sided shift on X. We set $Y = (0, 1)$ with $\mathfrak{F}$ the Borel σ-algebra and the Lebesgue measure λ. Let R denote the standard invertible odometer on Y. Omitting well-known details, we remove the dyadic rationals from Y.

To complete the example according to the construction in Section 2.1, we need to choose a generating partition for R. This will be a two set partition $\mathcal{Q} = \{Q_0, Q_1\}$, $\lambda(Q_i) = \frac{1}{2}$, which will be specified later. Accordingly, we set $\beta_0 = \{\frac{1}{2}, \frac{1}{2}\}$ and $\beta_1 = \{\frac{1}{3}, \frac{2}{3}\}$.

Our goal is to prove that the maximal automorphic factor of $T \times R$ on $X \times Y$ is R, using Corollary 26. In particular, we will show that for λ almost all points $y \in Y$ the associated measures ρ_y have disjoint carriers – that is, after removing a universal set of μ measure zero from $X \times Y$, for any pair of distinct points $y, y' \in Y$, the measures ρ_y and $\rho_{y'}$ have disjoint carriers when viewed as measures on X. (It is simple to show that the measures ρ_y on X are pairwise singular; this however is not enough to insure that the projection from $X \times Y$ to X is one-to-one.) Our technique is to first remove a set of fibers over certain points in Y, then remove sets from the remaining fibers using an inductive argument. All sets removed will have μ measure 0.

3.1. Removing a set of λ measure zero from Y

We begin by removing all nongeneric fibers from $X \times Y$. Actually we remove some sets of the form $X \times \{y\}$ by taking out all fibers which are not generic simultaneously for R, $R^2, R^4, \ldots, R^{2^p}, \ldots$.

We start with the following basic result.

Lemma 31. *The set*

$$F_0 = \{y \in Y \mid \lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} \chi_{Q_0}(R^i y) = \tfrac{1}{2}\}$$

is an R invariant set of λ measure one.

Proof. This follows immediately from the Birkhoff ergodic theorem. □

This implies that the following is true.

Lemma 32. *For all $y \in F_0$ the measure ρ_y 'sees' β_0 half the time and β_1 the other half.*

Proof. The proof follows from the general construction since the product measure ρ_y is determined by the symbolic coding of the point y which is half zeros and half ones for a generic point. □

We next apply the Birkhoff Ergodic Theorem to the transformations $R^2, R^4, R^8, \ldots$. Even though these are no longer ergodic, the averages still exist. For example, R^4 has four ergodic components $(0, \frac{1}{4}), (\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4}), (\frac{3}{4}, 1)$. We apply the ergodic theorem on each component, and remove four sets of measure zero from Y; we call the set of remaining points F_4.

We then take the intersection $\cap_{i=0}^{3} R^i F_4$ to obtain an invariant set of measure one. This set not only has the 'correct' average along y_i, it has the correct averages along $y_{4i}, y_{4i+1}, y_{4i+2}$ and y_{4i+3}.

We repeat this process inductively for each $R^{2^p}, p = 0, 1, 2 \ldots$, so that each of the following sets has measure one:

$$F_p = \cup_{q=0}^{2^p-1} F_{p,q} = \cup_{q=1}^{2^p-1} \{y \in Y \mid \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \chi_{Q_0}(R^{2^p i+q} y)$$
$$= \lambda(Q_0 \cap [\frac{q}{2^p}, \frac{q+1}{2^p}]) \cdot 2^p\}$$

To obtain an R invariant set, we take $\cap_{i=0}^{2^p-1} R^i F_p$, and then we take the intersection over p.

This gives us the following result.

Lemma 33. *There exists an R invariant set F of measure one in Y so that for all $y \in F$, along the sequences $2^p i + q, p = 0, 1, \ldots, q = 0, 1, \ldots 2^p - 1, i = 0, 1, \ldots,$ we obtain the correct ergodic averages for y as above.*

Proof. We define $F = \cap_{p=0}^{\infty} \cap_{i=0}^{2^p-1} R^i F_p$. □

Our first step is concluded by removing from $X \times Y$ the set $X \times F^c$ which is of μ measure zero. From now on, when we say 'y generic' we mean $y \in F$.

We are now ready to define the partition Q for the odometer which we use in the construction.

3.2. The adding machine and a two-set generator

We briefly recall the cutting and stacking description of the adding machine transformation. The transformation R is defined as the standard transformation 'going up' the stacks linearly.

We start with the unit interval, viewed as a column of height 1. At stage 1, we cut the preceeding column in half and stack the right hand side on top of the left resulting in a column of height 2. The transformation, as usual, goes 'up' the column. At stage n, we start with a column of height 2^{n-1}, cut it in half and stack the right hand side onto the left hand side resulting in a column of height 2^n.

A two set partition Q for the adding machine on the unit interval which is a one-sided generator is defined by:

$$
\begin{aligned}
Q_0 &= \cup_{k=0}^{\infty}\left(\frac{2^k-1}{2^k}, \frac{2^k-1}{2^k}+\frac{1}{2^{k+2}}\right) \\
&= (0, 1/4) \cup (1/2, 5/8) \cup (3/4, 13/16) \cup \ldots \\
Q_1 &= \cup_{k=0}^{\infty}\left(\frac{2^k-1}{2^k}+\frac{1}{2^{k+2}}, \frac{2^{k+1}-1}{2^{k+1}}\right) \\
&= (1/4, 1/2) \cup (5/8, 3/4) \cup (13/16, 7/8) \cup \ldots
\end{aligned}
$$

We explain how the partition was chosen in order to show it is a generator. Writing the space as:

(1) $$Y = (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}) \cup (\frac{3}{4}, \frac{7}{8}) \ldots$$

we will refer to the above subintervals as 'pieces'.

Each piece is a level of one of the stacks of the adding machine. Each of these levels is cut in half at the respective next stage. As each piece is cut in half, the left side is put into Q_0 and the right side is put into Q_1. For example, the first piece $(0, \frac{1}{2})$, a level of stage 1, is cut at the second stage into two quarters $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2})$. The second piece is cut into two eighths at the third stage. In general, the i^{th} piece is cut into two $1/2^{i+2}$ dyadic subintervals of Y at the $(i+1)^{st}$ stage of the cutting and stacking process.

This continues inductively and we obtain the following.

Lemma 34. *For every $n \geq 2$, in the column of height 2^n, exactly $2^{n-1} - 1$ levels (an odd number) are contained in Q_0; exactly $2^{n-1} - 1$ levels are contained in Q_1; the remaining two levels consist evenly of subpieces of Q_0 and Q_1.*

3.3. The Q-names for $y \in Y$

We now examine the Q-names of points in Y in order to prove that Q is a generator. Since the two sets Q_i are of measure $\frac{1}{2}$ we have that for each $y \in Y$, the Q-name $y = (y_0 y_1 y_2 \ldots)$ consists, on average, of half 0's and half 1's. However, depending on the exact location of y, we can say more about the Q-name. Furthermore, we show that by knowing the Q-name of a point y along various subsequences of integers, we can determine the precise location of y in Y. This will show that Q is a generator.

Suppose first that $y \in (0, 1/4)$. We consider the sequence $y_0, y_4, y_8, y_{12}, \cdots$. Since $R^{4i}(y)$, $i \in \mathbb{N}$, are all in $(0, 1/4)$, it follows that $y_{4i} = 0$ for each i. One consequence of this fact is that all the measures ρ_y for $y \in (0, 1/4)$ 'see' the measure $\beta_0 = (1/2, 1/2)$ along the subsequence $0, 4, 8, 12, \cdots$.

Let $y \in (1/4, 1/2)$, and again look at the sequence $y_0, y_4, y_8, y_{12}, \cdots$. All points $R^{4i}(y)$ are in $(1/4, 1/2)$ and hence $y_{4i} = 1$. Then for these y, all the measures ρ_y 'see' the measure $(1/3, 2/3)$ along the subsequence $0, 4, 8, 12$.

If $y \in (1/2, 3/4)$, the sequence $R^{4i}y$ is always in $(1/2, 3/4)$ but alternates between $(1/2, 5/8)$ and $(5/8, 3/4)$. Hence, the Q-name alternates between 0 and 1 along the sequence y_{4i}. One can calculate what this means for the ρ_y measures in terms of the β_0 and β_1's.

Finally, let $y \in (3/4, 1)$. To see the sequence $y_0, y_4, y_8, y_{12}, \cdots$, we note that the sequence $R^{4i}y$ is always in $(3/4, 1)$. The relative proportion of Q_0 and Q_1 in $(3/4, 1)$ is half-and-half, and so along the sequence $0, 4, 8, 12, \cdots$, y_{4i} is 0 half the time and 1 half the time.

From the above discussion, it is clear that by looking at the Q-name y_i along the subsequence $i = 0, 4, 8, \cdots$ we can distinguish points in $(0, \frac{1}{4})$ from points in $(\frac{1}{4}, \frac{1}{2})$. This idea is refined in the next lemma to determine what quarter y is in. For ease of notation we use the symbol y_i to denote both the point $R^i y$ and its symbolic coding.

Lemma 35. *For any generic y, by looking at the four sequences $\{y_{4i+j}\}_{i \geq 0}$, for $j = 0, 1, 2, 3$, we can determine which quarter y is in.*

Proof. Suppose that $y \in (0, \frac{1}{4})$, and consider the sequence $y_2, y_6, y_{10}, y_{14}, \cdots$. The points are all in $(\frac{1}{4}, \frac{1}{2})$, so each symbol $y_{4i+2} = 1$.

We next look at the sequence $y_1, y_5, y_9, y_{13}, \cdots$. These points are in $(\frac{1}{2}, \frac{3}{4})$, and alternate between the sets Q_0 and Q_1.

Looking now at the sequence $y_3, y_7, \cdots$, we see 0 half the time and 1 half the time – but not alternating.

Similar reasoning shows that for each generic y we will see one of the following 4 patterns repeated in the Q-name:

$0a1b$

$a1b0$

$1b0a$

$b0a1$

where a, b are either 0 or 1. (The a alternate between the two, while the pattern for b is slightly more complicated.)

Each distinct pattern corresponds to a different quarter of the interval. We note that there is only one pattern and three distinct cyclic rotations of it. This proves the lemma. □

To continue the inductive step, we look at the eight sequences $\{8i + j\}_{i \geq 0}$ for $j = 0, 1, 2, 3, 4, 5, 6, 7$. In this case, we have the pattern $001a011b$ and its rotations appearing in $y_i y_{i+1} \cdots y_{i+7}$. From this we obtain the following result.

Lemma 36. *For a generic point y, by looking at the eight sequences $\{y_{8i+j}\}_{i \geq 0}$, for $j = 0, 1, 2, \cdots, 7$, we can determine which eighth contains the point y.*

An inductive argument gives the general result.

Lemma 37. *For a generic y, using the k^{th} piece in formula 1, and by looking at the 2^{k+1} sequences $\{y_{2^{k+1}i+j}\}_{i \geq 0}$, for $j = 0, 1, 2, \cdots, 2^{k+1} - 1$, we can determine which 2^{k+1} dyadic subinterval contains y.*

The fact that each pattern is distinct under rotation follows because at each stage there are an odd number of levels in Q_0 (and the same odd number in Q_1).

Since the dyadic intervals generate the point partition in Y, we have proved the following.

Lemma 38. *Q is a generator for R.*

3.4. The inductive construction of a carrier for μ

We now turn our attention to the product space. We have a binary coding for each coordinate of any point $(x, y) \in X \times Y$; x_i is 0 or 1 depending on the dyadic expansion of X, while y_i depends on the partition Q of Y. The probability that $x_i = 1$ is $\frac{1}{2}$ if $y_i = 0$ because then $\beta_i = \beta_0 = \{\frac{1}{2}, \frac{1}{2}\}$. Similarly the probability that $x_i = 1$ is $\frac{2}{3}$ if $y_i = 1$ because then $\beta_i = \beta_1 = \{\frac{1}{3}, \frac{2}{3}\}$.

We combine these facts with the specific form of y-names along subsequences of length 2^p for $p = 2, 3, \ldots$ to construct a countable number of carriers for μ inductively. At the p^{th} stage we refine the previous carrier along dyadic subintervals of length 2^{-p}.

This section is organized as follows. Starting with points in $X \times F$, we first define the set A_0 consisting of those (x, y) whose x coordinates are generic for the shift with respect to ρ_y. We start the inductive argument on R^4; i.e., or $p = 2$. In Section 3.3 we saw that points in distinct quarters of $(0, 1)$ have distinct coordinate patterns along the subsequences $\{y_{4i}\}$, $\{y_{4i+1}\}$, $\{y_{4i+2}\}$, and $\{y_{4i+3}\}$. These, in turn, induce 4 distinct families of ρ_y measures, one for each quarter. By looking at points in X which are generic for each of those four families of measures, we induce four disjoint sets on X coming from the fiber measure carriers.

We then give the general inductive argument. At the p^{th} stage, we will have 2^p distinct carriers for fiber measures ρ_y projecting onto 2^p disjoint sets in X.

We begin with the following result.

Proposition 39. *The set*

$$A_0 = \{(x, y) \mid \lim \frac{1}{n+1} \sum_{0}^{n} x_i = \frac{\frac{1}{2} + \frac{2}{3}}{2} = \frac{7}{12}, y \in F\}$$

is a $T \times R$ invariant set of μ measure one. That is, it is a carrier of the measure μ.

Proof. The invariance of A_0 is clear; applying $(T \times R)^{-1}$ just adds an x_0 term which will not affect the average if it exists.

We prove that the measure of A_0 is 1 by showing that for each y, $\rho_y(A_0) = 1$.

Given a fixed generic $y = (y_0 y_1 y_2 \cdots)$ there are two disjoint subsequences of indices $\mathbb{I} \cup \mathbb{J} = \mathbb{N}$, where $\beta_i = \beta_0$ for $i \in \mathbb{I}$ and $\beta_j = \beta_1$ for $j \in \mathbb{J}$. By the definition of the measure ρ_y almost all points in $X \times \{y\}$ satisfy:

$$\lim \frac{1}{\# I_n} \sum_{i \in I_n} x_i = \frac{1}{2},$$

where $I_n = I \cap \{0, 1, 2 \dots, n\}$ and $\# I_n$ is its cardinality.
We also have for almost all points

$$\lim \frac{1}{\# J_n} \sum_{j \in J_n} x_i = \frac{2}{3}.$$

Therefore, by throwing out the two sets of measure zero (the sets for which the above limits do not exist) we get the resulting set is of ρ_y measure 1. We also note that the set is contained in A_0, but the set A_0 could contain more points from the fiber ρ_y. This proves that $\mu(A_0) = 1$. □

The basic idea of the inductive step in the construction is to refine the set A_0 until we have disjoint carriers for the measures ρ_y.

The first inductive step: choosing a carrier with good R^4 averages

We start the induction with $p = 2$ by constructing a carrier for μ which induces 4 disjoint ρ_y carriers on X.

We recall that $(0, \frac{1}{4}) \subset Q_0$ and $(\frac{1}{4}, \frac{1}{2}) \subset Q_1$, and these two sets are invariant under R^4, as are the images of the two sets under R.

We now construct a subset of the carrier obtained above, denoted $A_2 \subseteq A_0 \subset X \times Y$.
We first define

$$A_{2,0} = \left\{ (x, y) \mid \begin{cases} 0 < y < \frac{1}{4}, y \in F \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i} = \dfrac{1}{2} \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i+1} = \dfrac{7}{12} \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i+2} = \dfrac{2}{3} \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i+3} = \dfrac{7}{12} \end{cases} \right\}$$

Then $(T \times R)(A_{2,0})$ is the following set

$$A_{2,1} = \left\{ (x, y) \mid \begin{cases} \frac{1}{2} < y < \frac{3}{4}, y \in F \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i} = \dfrac{7}{12} \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i+1} = \dfrac{2}{3} \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i+2} = \dfrac{7}{12} \\ \lim\limits_{n \to \infty} \dfrac{1}{n+1} \sum\limits_0^n x_{4i+3} = \dfrac{1}{2} \end{cases} \right\}$$

For $A_{2,2} = (T \times R)^2(A_{2,0})$ we get the averages in order

$$\frac{2}{3} \quad \frac{7}{12} \quad \frac{1}{2} \quad \frac{7}{12}$$

And $A_{2,3} = (T \times R)^3(A_{2,0})$ shifts them around once more. The fourth image is sent back to $A_{2,0}$.

Lemma 40. *$A_2 = A_{2,0} \cup A_{2,1} \cup A_{2,2} \cup A_{2,3} \subseteq A_0$ is a $T \times R$ invariant set of measure one.*

Lemma 41. *The set A_2 induces carriers of ρ_y for each generic $y \in Y$ with the property that for all points $y, y' \in F$, if y and y' are in different quarters of Y, then the two associated measures have strictly disjoint induced carriers, considered as measures on X.*

Proof. Since y and y' are in different quarters, using the sets $A_{2,q}$ above, we will see different averages for the x_{4i+q} sequences. We will call the disjoint sets on X induced by the above carriers $C_{2,0}$, $C_{2,1}$, $C_{2,2}$ and $C_{2,3}$. □

The p^{th} inductive step: choosing a carrier with good R^{2^p} averages

We assume now that we are given a carrier A_{p-1} for μ which induces carriers for the measures ρ_y, which, as measures and sets on X are disjoint for all generic y's in distinct intervals of the form $(q/2^{p-1}, (q+1)/2^{p-1}), q = 0, \ldots, 2^{p-1} - 1$. Using the notation given earlier, we denote each carrier by $C_{p,q} \times \{y\}$, if $y \in (q/2^{p-1}, (q+1)/2^{p-1})$ with $C_{p,q} \subset X$. Our assumption then is that if $q \neq r$, then $C_{p,q} \cap C_{p,r} = \emptyset$. Furthermore we assume that the points $x \in C_{p,q}$ 'see' the correct average along the sequences of the form: $\{x_{2^{(p-1)}i+q}\}_i$ for each $q = 0, \ldots 2^{p-1} - 1$.

We shrink the carrier set A_{p-1} to A_p as follows. Define

$$A_{p,q} = \left\{ (x,y) \mid \begin{cases} \frac{q}{2^p} < y < \frac{q+1}{2^p}, y \in F \\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} x_{2^p i} = \alpha_0 \\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} x_{2^p i+1} = \alpha_1 \\ \vdots \\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} x_{2^p i+q} = \alpha_q \\ \vdots \\ \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} x_{2^p i+2^p-1} = \alpha_{2^p-1} \end{cases} \right\}$$

where the α_q are the appropriate averages determined by the y name along the sequence $y_{2^p i - q}$. In particular, $2^{p-1} - 1$ of the α_q are $1/2$; $2^{p-1} - 1$ are $2/3$; the last two α_q are $7/12$. As before, they form into 2^p patterns cyclicly rotated. We therefore have that points y, y' in different dyadic intervals of length 2^p have strictly disjoint carriers for their corresponding ρ_y measures.

We define the set $A_p = \cup_{q=0}^{2^p-1} A_{p,q}$.

Theorem 42. *The set A_p is a $T \times R$ invariant set of measure 1. The carriers for ρ_y and $\rho_{y'}$ derived from A_p are disjoint for y, y' in different 2^pth subintervals.*

Finally, the set $A = \cap_{p\geq 0}^{\infty} A_p$ provides a carrier of μ satisfying the hypotheses of Theorem 25.

3.5. The main result

Theorem 43. *The example defined above gives an ergodic invariant measure σ for a one-sided 2-state shift T on X_2^+ with a maximal automorphic factor isomorphic to the dyadic adding machine.*

The same arguments can be used to obtain the following result.

Theorem 44. *Given any n and the n-odometer transformation R on Y there is a measure σ on X_n^+ such that the resulting shift is ergodic, preserves σ, and has R on Y as its maximal automorphic factor.*

Remark. It is easy to see that the general construction applied to an arbitrary zero entropy R transformation has λ almost all pairs ρ_y, $\rho_{y'}$ mutually singular. However, it is not known whether or not the projection of $X \times Y$ onto $X \times Y$ will always be invertible.

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