Northeastern University

Alpha-Type and Exhaustive Weakly Wandering Sequences for Ergodic Infinite Measure Preserving Transformations

2007-2008

Throughout this talk

$(T, X, \mathcal{B}, \mu)$

- $(X, \mathcal{B}, \mu)$: σ-finite, nonatomic, Lebesgue Space
- $\mu(X) = \infty$
- T: Ergodic, Invertible, Measure-Preserving Transformation

Definition

- $\{n_i : i = 0, 1, 2, \cdots\}$ a sequence of integers ($n_i \neq n_j$ for $i \neq j$)
- W a measurable set in X

are **Exhaustive Weakly Wandering** for T if

$$X = \bigcup_{i=0}^{\infty} T^{n_i} W \quad (Exh.)$$

$$T^{n_i} W \cap T^{n_j} W = \emptyset, i \neq j \quad (W.W.)$$

$\{n_i\}$ is the **Ex.W.W. Sequence**.

W is the **Ex.W.W. Set**.

Theorem (Jones - Krengel)

Every ergodic, infinite measure preserving transformation has Exhaustive Weakly Wandering sets and sequences.

However, for many (most) transformations no Exhaustive Weakly Wandering set or sequence is known.

Definition

T is of α-**type**, for $0 \leq \alpha \leq 1$, if

$$\limsup_{n \to \infty} \mu(T^n A \cap A) = \alpha \cdot \mu(A)$$

for all A satisfying $\mu(A) < \infty$.

To date, the only known Alpha-types for which specific Exhaustive Weakly Wandering sequences and sets are explicitly known are for

$$\alpha = \frac{n-1}{n}$$

To date, no Exhaustive Weakly Wandering sequence is known to work for two different Alpha-Types.

Main Theorem

Theorem

There exists a sequence $\mathbb{B} = \{b_0, b_1, \cdots\}$. $b_i \neq b_j$ *for* $i \neq j$ *such that there exists a family of maps* T_α, $0 \leq \alpha \leq 1$ *so that for all* α

- T_α *is ergodic infinite measure preserving*
- $\mathbb{B}$ *is an Exhaustive Weakly Wandering sequence for all* T_α
- *All the Exh.W.W. sets* W_α *have measure one*
- T_α *is of* α*-type*

The Sequence $\mathbb{B}$ is a Direct Sum

The sequence $\mathbb{B}$ is a **Direct Sum** (Finite Sum of Finite Sets)

$$\mathbb{B} = \oplus_{n=1}^{\infty} \mathbb{B}_n$$

$\mathbb{B}_1 = \{0, 2\}$
$\mathbb{B}_2 = \{0, 16, 32, 48, 64, 80, 96, 112\}$
$\mathbb{B}_3 = \{0, 1024, 2048, 3072, 4096, 5120 \cdots 130048\}$
$\vdots$
$\mathbb{B}_k = \{0, b_k, 2 \cdot b_k, 3 \cdot b_k, \cdots, (2^{2^k - 1} - 1) \cdots b_k\}$

where $b_k = 2^k \cdot \prod_{j=0}^{k-1} 2^{2^j + j - 1}$

Ohio-State - An Example of Hajian-Kakutani

The fundamental ideas of the proofs can be illustrated with a simple example.

In fact it is the first known example for the two concepts of Alpha-type and Exhaustive Weakly Wandering.

Specifically,

- Alpha-type for $\alpha = \frac{1}{2}$
- The Ex.W.W. set is displayable and has measure 1
- The Ex.W.W. sequence is definable and is

$$\mathbb{B} = IP\{2^{2n+1}\} = \{0, 2, 8, 10, \cdots\}$$

0 *corresponds to the empty sum*

The example will be presented Three ways.

1. a Cutting, Stacking, Spacer construction
2. a Skyscraper construction
3. an Adding Machine construction

1. The Cutting, Stacking, Spacer construction will allow the Alpha-type to be be controlled.
2. The Skyscraper construction will allow the Exhaustive Weakly Wandering set to be displayed.
3. The Adding Machine construction will allow control of the Exhaustive Weakly Wandering sequence

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Construction - Cut, Add Spacers, Stack

Stage 1: Start with the Interval $[0, 1)$

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Cut in half

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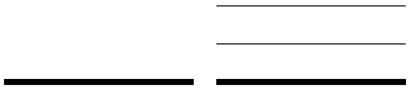
Add 2 spacers - on Right Column



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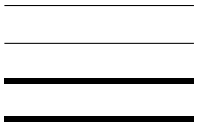

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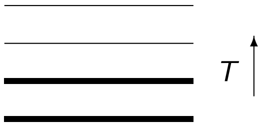
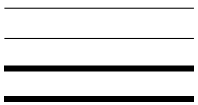
The Transformation T goes up



Construction - Cut, Add Spacers, Stack

Stage 2: Start with previous column



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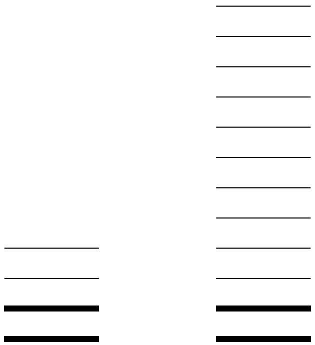
Add 2^3 spacers - on Right Column



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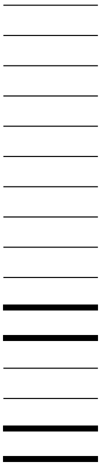

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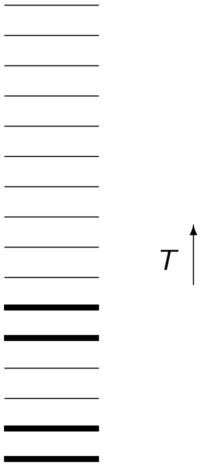
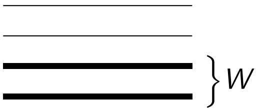


n^{th} Stage of Constuction

1. Cut previous column in half.
2. Add 2^n spacers on right subcolumn.
3. Stack right column on top of left column.
4. T defined going up the column.

$W = [0, 1)$ is the Exhaustive Weakly Wandering Set

At Stage 1 we can "see"
W and $T^2 W$ "filling" the space



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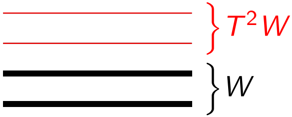

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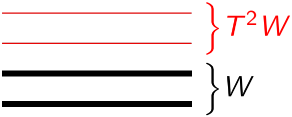
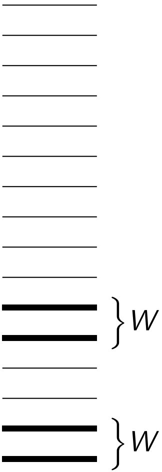


At Stage 2, W, T^2W and T^8W $T^{10}W$ "fill"



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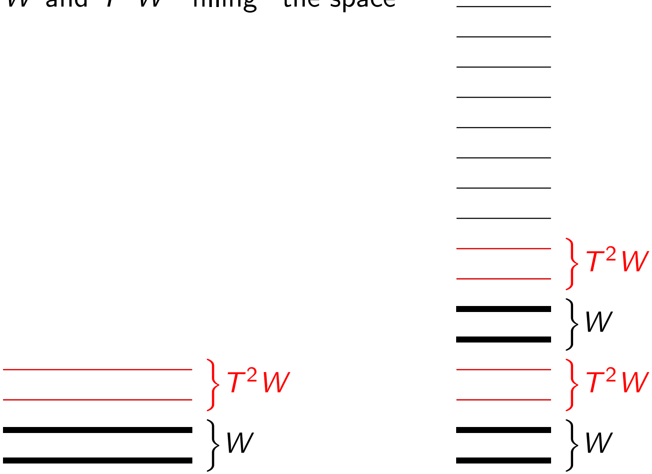

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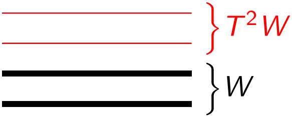

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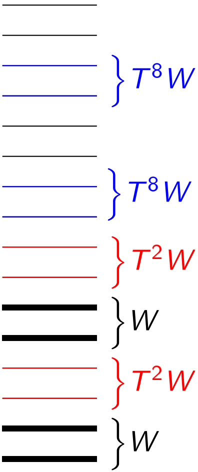

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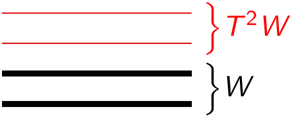

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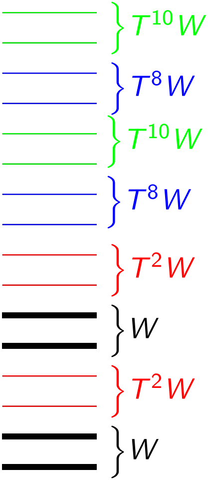


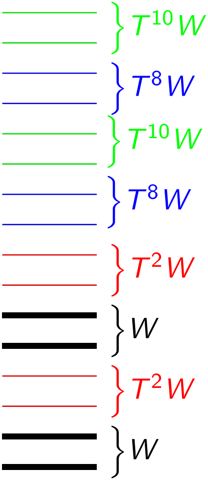
Exhaustive Weakly Wandering Sequence for Ohio-State

So far the Ex.W.W. sequence for the Ohio-State is

$$\{0, 2, 8, 10\}$$

This can be derived as

$$\{0, 2\} \oplus \{0, 8\}$$



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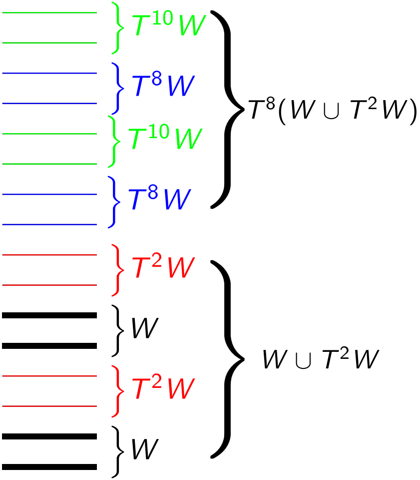

Exhaustive Weakly Wandering Sequence for Ohio-State

The Exhaustive Weakly Wandering Sequence for the Ohio State Example is a Direct Sum.

$$\bigoplus_{n=1}^{\infty}\{0, 2^{2n-1}\} = \bigoplus_{n=0}^{\infty} 2^{2n}\{0, 2\}$$

2^{2n-1} is the number of spacers being added at the n^{th} stage.

Alpha $= \frac{1}{2}$ for the Ohio State example

We illustrate on $W = [0, 1)$

$$m(W \cap T^{2^{2n}} W) = \frac{1}{2}$$

Otherwise

$$m((W \cap T^k W) < \frac{1}{2}$$

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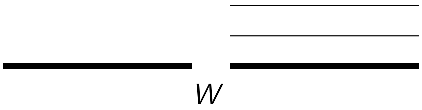

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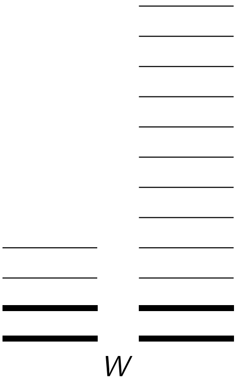
Stage 2



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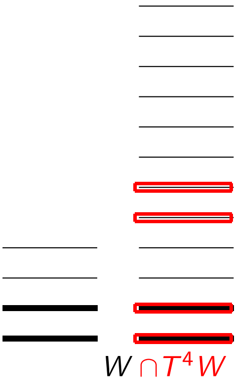


Stage 1

General Construction

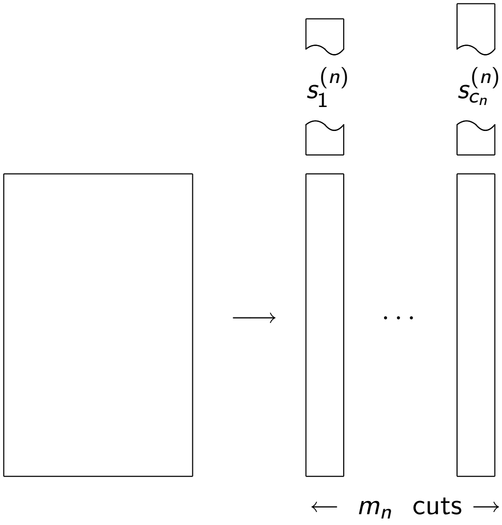
For the general constuction we only need to define a sequence of Cuts and a sequence of Spacers to place atop the sub-columns before stacking (right on top of left).

- The Number of Cuts

$$m_n \geq 2$$

- The Number of Spacers

$$\mathcal{S}_n = \{s_1^{(n)}, \cdots, s_{c_n}^{(n)}\}$$



Theorems - Sufficiency for Alpha-Types

Theorem

If in the general construction (described above)

- $m_k > m_{k-1}$, *the cuts are increasing*
- $s_{n+1}^{(k)} \geq \sum_{i=1}^{n} s_i^{(k)} + n \cdot h_{k-1}$, $n \geq 1$

Then T is of 0*-type.*

The first spacer $s_1^{(k)} \geq 0$ is arbitrary - in practice it will be set to 0.

Theorem

If in the general construction

- $m_k > m_{k-1}$
- $n_k < m_k$ *such that* $\lim_{k\to\infty} \frac{n_k}{m_k} = \alpha$
- $s_i^{(k)} = 0$ *for* $1 \le i \le n_k$
- $s_{j+1}^{(k)} \ge \sum_{i=1}^{j} s_i^{(k)} + j \cdot h_{k-1},\ j \ge n_k$

Then T *is of* α*-type*

Note, the first n_k need only be equal. We set them to 0 for simplicity.

The first theorem follows from the second - but all the concepts can be seen in the proof of the first.

Adding Machine Construction

If we don't stack the columns, we see a Skyscraper.

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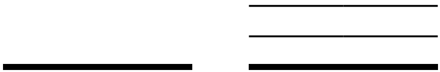

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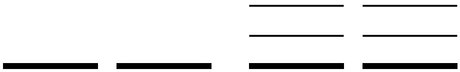

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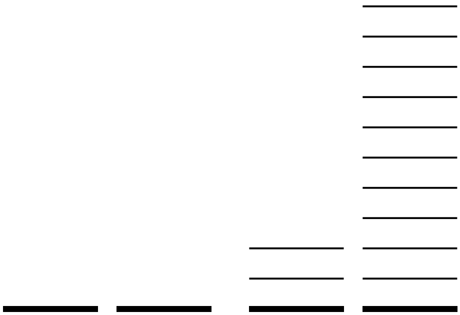
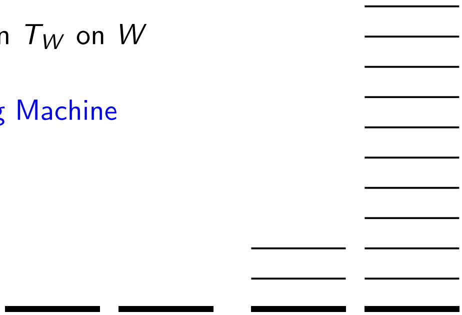


Adding Machine Construction

If we don't stack the columns, we see a Skyscraper.

$W = [0, 1)$ is the Base.
The Induced Transformation T_W on W

"is" the Dyadic Adding Machine



Ohio State - Reconstructed

$$\Omega = \prod_{i=0}^{\infty} \{0,1,2,3\}$$

$\tau =$ " $+1$" (carry to the right).

$W = \{\omega \in \Omega : \omega_i \in \{0,1\}\} \subset \Omega$
$X = \{\omega : \exists N,\ \omega_i \in \{0,1\}\ \forall i \geq N\}$

$$\begin{aligned}
\omega &= (000\cdots) \in W \\
\tau\omega &= (100\cdots) \in W \\
\tau^2\omega &= (200\cdots) \\
\tau^3\omega &= (300\cdots) \\
\tau^4\omega &= (010\cdots) \in W \\
\tau^5\omega &= (110\cdots) \in W \\
\tau^6\omega &= (210\cdots) \\
&\vdots
\end{aligned}$$

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