THE NOVIKOV-BOTT INEQUALITIES

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Abstract. We generalize the Novikov inequalities for 1-forms in two different directions: first, we allow non-isolated critical points (assuming that they are non-degenerate in the sense of R.Bott), and, secondly, we strengthen the inequalities by means of twisting by an arbitrary flat bundle. We also obtain an L^2 version of these inequalities with finite von Neumann algebras.

The proof of the main theorem uses Bismut's modification of the Witten deformation of the de Rham complex; it is based on an explicit estimate on the lower part of the spectrum of the corresponding Laplacian.

LES INEGALITES DE NOVIKOV-BOTT

Résumé. Nous généralisons les inégalités de Novikov pour les 1-formes dans deux directions différentes: tout d'abord, nous permetons des points critiques non isolés (tout en supposant qu'ils sont non-dégénérés au sens de R. Bott), et ensuite, nous renforçons les inégalités au moyen d'un croisement par un fibré vectoriel arbitraire. Nous obtenons également une L^2-généralisation de ces inégalités pour des algébres von Neumann finies.

La preuve du théorme principal utilise la modification de Bismut de la déformation du complexe de Rham; elle est basée sur une estimation explicite de la partie inférieure du spectre du Laplacien correspondant.

1. **Version française abrégée.** Dans [N1], S.P.Novikov associe á toute classe de cohomologie réelle $\xi \in H^1(M,\mathbb{R})$ une suite de nombres $\beta_0(\xi),,\dots,\beta_n(\xi)$ et prouve que pour toute 1-forme fermée ω sur M, ayant des points critiques non dégénérés, les inégalités suivantes ont lieu:

$$m_p(\omega) \geq \beta_p(\xi), \qquad p = 0,1,2,\dots$$

Ici $\xi = [\omega] \in H^1(M,\mathbb{R})$ est la classe de cohomologie de ω et $m_p(\omega)$ dénote le nombre de points critiques de ω d'indice de Morse p.

Dans cet article nous généralisons les inégalités de Novikov pour les 1-formes dans deux directions différentes: tout d'abord, nous permetons des points critiques non isolés, et ensuite, nous renforçons les inégalités au moyen d'un croisement par un fibré vectoriel arbitraire. Nous considérons aussi des fibrés dont les fibres sont des modules sur des algébres von Neumann finies.

Soit M une variété lisse fermée munie d'un fibré vectoriel complexe plat fixé F. Soit ω une 1-form sur M, lisse fermée á valeurs réelles, qui est non dégénérée au sens de Bott. Pour chaque composante connexe Z de l'ensemble C des points critiques de ω nous construisons (Section 4) le *polynôme de Poincaré* $P_{Z,F}(\lambda)$ de composante Z que nous utilisons pour définir le *polynôme de comptage de Morse* $M_{\omega,F}(\lambda) = \sum_Z \lambda^{\mathrm{ind}(Z)} P_{Z,F}(\lambda)$, oú la somme est prise sur toutes les composantes connexes Z de C et $\mathrm{ind}(Z)$ est l'indice de Z défini dans la Section 4. Definissons le

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polynôme de Novikov par $N_{\xi,F}(\lambda) = \sum_{i=0}^{n} \lambda^i \beta_i(\xi, F)$, oú les $\beta_i(\xi, F)$ sont les nombres de Novikov généralises, cf. Section 2.

Théorème . *Il existe un polynôme $Q(\lambda)$ á coefficients entiers non-negatifs, tel que:*

$$M_{\omega,F}(\lambda) \; - \; N_{\xi,F}(\lambda) \; = \; (1+\lambda)Q(\lambda).$$

Notre preuve du théorème ci-dessus est basée sur légère modification de la déformation de Witten [Wi] suggérée par Bismut [Bi] dans sa preuve des inégalités de Morse dégénereés de Bott. Cependant notre preuve est assez différente de celle de [Bi], même dans le cas $[\omega] = 0$. Nous éntons tout à fait l'analyse probabiliste des noyaux de chaleur, qui est la partie la plus difficile de [Bi]. Au lieu de ceci, nous donnons une estimation explicite du nombre de valeurs propres "petites" du Laplacien déformé.

Helffer et Sjöstrand [HS] donnent une preuve analytique très élégante des inégalités de Morse dégénérées de Bott. Bien qu'ils utilisent aussi les ideées de [Wi], leur méthode est complétement différente de celle de [Bi] et de la nôtre. Il n'est pas clair si cette méthode peut être appliquée au cas $\xi \neq 0$.

Si tous les zéros de ω sont non-dégénérés, le théorème ci-dessus donne les inégalités

$$\sum_{i=0}^{p}(-1)^i m_{p-i}(\omega) \;\geq\; d^{-1}\cdot\sum_{i=0}^{p}(-1)^i\beta_{p-i}(\xi,F), \qquad p=0,1,2,\dots,$$

oú $d = \dim F$ et $m_p(\omega)$ dénote le nombre de points critiques de ω d'indice p. Les dernière inégalités coïncident avec les inégalités de Novikov [N1] dans le cas spécial oú $F = \mathbb{R}$ avec la structure plate triviale.

2. **The generalized Novikov numbers.** Let M be a closed manifold and let F be a complex flat vector bundle over M. We will denote by $\nabla : \Omega^\bullet(M,F) \to \Omega^{\bullet+1}(M,F)$ the covariant derivative on F. Given a closed 1-form $\omega \in \Omega^1(M)$ on M with real values, it determines a family of connections on F (the *Novikov deformation*) parameterized by the real numbers $t \in \mathbb{R}$

$$\nabla_t : \Omega^i(M,F) \to \Omega^{i+1}(M,F), \qquad \nabla_t :\ \theta \mapsto \nabla\theta + t\omega\wedge\theta. \tag{0.1}$$

All the connections ∇_t are flat, i.e. $\nabla_t^2 = 0$. Denote by F_t the flat vector bundle defined by the connection (0.1). Note that changing ω by a cohomologious 1-form determines a gauge equivalent connection ∇_t and so the cohomology $H^\bullet(M,F_t)$ depends only on the cohomology class $\xi = [\omega] \in H^1(M,\mathbb{R})$ of ω. One can show that there exists a *finite* subset $S \subset \mathbb{R}$ (the set *jump points*) such that $\dim H^i(M,F_t)$ is constant for $t \notin S$ and it jumps up for $t \in S$. The dimension of $\dim H^i(M,F_t)$ for $t \notin S$ is called the i-th *(generalized) Novikov number* $\beta_i(\xi,F)$.

3. **Assumptions on the 1-form.** Let C denote the set of critical points of ω (i.e. the set of points of M, where ω vanishes). We assume that ω is *non-degenerate in the sense of Bott*, i.e. C is a submanifold of M and that the Hessian of ω is a non-degenerate quadratic form on the normal bundle $\nu(C)$ to C in M. Here by the Hessian of ω we understand the Hessian of the unique function f defined in a tubular neighborhood of C and such that $df = \omega$ and $f_{|_C} = 0$.

4. **The main result.** Let Z be a connected component of the critical point set C and let $\nu(Z)$ denote the normal bundle to Z in M. Since the Hessian of ω is non-degenerate, the bundle $\nu(Z)$ splits into the Whitney sum of two subbundles $\nu(Z) = \nu^+(Z) \oplus \nu^-(Z)$, such that the Hessian is strictly positive on $\nu^+(Z)$ and strictly negative on $\nu^-(Z)$. The dimension of the bundle $\nu^-(Z)$ is called the *index* of Z (as a critical submanifold of ω) and is denoted by $\operatorname{ind}(Z)$. Let $o(Z)$ denote the *orientation bundle of $\nu^-(Z)$, considered as a flat line bundle.* Consider the *twisted Poincaré polynomial* of Z

$$P_{Z,F}(\lambda) \;=\; \sum \lambda^i \dim_{\mathbb{C}} H^i(Z, F|_Z \otimes o(Z)) \tag{0.2}$$

(here $H^i(Z, F|_Z \otimes o(Z))$ denote the cohomology of Z with coefficients in the flat vector bundle $F|_Z \otimes o(Z))$ and define using it the following *Morse counting polynomial*

$$M_{\omega,F}(\lambda) \;=\; \sum_Z \lambda^{\operatorname{ind}(Z)} P_{Z,F}(\lambda), \tag{0.3}$$

where the sum is taken over all connected components Z of C.

With one-dimensional cohomology class $\xi = [\omega] \in H^1(M, \mathbb{R})$ and the flat vector bundle F, one can associate the *Novikov polynomial*

$$N_{\xi,F}(\lambda) \;=\; \sum_{i=0}^{n} \lambda^i \beta_i(\xi, F), \qquad n = \dim M. \tag{0.4}$$

main There exists a polynomial $Q(\lambda)$ with non-negative integer coefficients, such that

$$M_{\omega,F}(\lambda) \;-\; N_{\xi,F}(\lambda) \;=\; (1+\lambda)Q(\lambda). \tag{0.5}$$

The main novelty in this theorem is that it is applicable to the case of 1-forms with non-isolated singular points. Thus, we obtain, in particular, a new proof of the degenerate Morse inequalities of R.Bott. Moreover, Theorem **??** yields a generalization of the Morse-Bott inequalities to the case of an arbitrary flat vector bundle F; this generally produces stronger inequalities as shown in Section 7.

Corollary 5 (Euler-Poincaré theorem)**.** *Under the conditions of Theorem* **??***, the Euler characteristic of M can be computed as $\chi(M) = \sum_Z (-1)^{\operatorname{ind}(Z)} \chi(Z)$, where the sum is taken over all connected components $Z \subset C$.*

6. **The case of isolated critical points.** Consider now the special case when all critical points of ω are isolated. Theorem **??** gives the inequalities

$$\sum_{i=0}^{p} (-1)^i m_{p-i}(\omega) \;\geq\; d^{-1} \cdot \sum_{i=0}^{p} (-1)^i \beta_{p-i}(\xi, F), \qquad p = 0, 1, 2, \ldots, \tag{0.6}$$

where $d = \dim F$ and $m_p(\omega)$ denotes the number of critical points of ω of index p. The last inequalities coincide with the Novikov inequalities [N1] when $F = \mathbb{R}$ with the trivial flat structure. Examples described in Section 7, show that using of flat vector bundles F gives sharper estimates in general.

On the other hand, (0.6) also generalizes the Morse type inequalities obtained by S.P.Novikov in [N2], using Bloch homology (which correspond to the case, when $[\omega] = 0$ in (0.6)).

7. **Examples.** Here we describe examples, where the Novikov numbers twisted by a flat vector bundle F (as defined above) give greater values (and thus stronger inequalities) than the usual Novikov numbers (where $F = \mathbb{R}$ or $\mathbb{C}$, cf. [N1, Pa]).

Let $k \subset S^3$ be a smooth knot and let the 3-manifold X be the result of $1/0$-surgery on S^3 along k. Note that the one-dimensional homology group of X is infinite cyclic and thus for any complex number $\eta \in \mathbb{C}$, $\eta \neq 0$, there is a complex flat line bundle F_η over X such that the monodromy with respect to the generator of $H_1(X)$ is η. By a choice of the knot k and the number $\eta \in \mathbb{C}^*$, we may make the group $H^1(X, F_\eta)$ arbitrarily large, while $H^1(X, \mathbb{C})$ is always one-dimensional.

Consider now the 3-manifold M which is the connected sum $M = X \# (S^1 \times S^2)$. Thus $M = X_+ \cup X_-$, where $X_+ \cap X_- = S^2$, $X_+ = X - \{disk\}$ and $X_- = (S^1 \times S^2) - \{disk\}$. Suppose F is a flat complex line bundle F over M such that its restriction to X_+ is isomorphic to $F_\eta|_{X_+}$. Consider the class $\xi \in H^1(X, \mathbb{R})$ such that its restriction onto X_+ is trivial and its restriction to X_- is the generator.

By using the Mayer-Vietoris sequence, we show that $\beta_1(\xi, F) = \dim_{\mathbb{C}} H^1(X, F_\eta)$. As we noticed above, this number can be arbitrarily large, while $\dim_{\mathbb{C}} H^1(M, \mathbb{C}) = 2$.

8. **Sketch of the proof of Theorem ??.** Our proof of Theorem **??** is based on a slight modification of the Witten deformation [Wi] suggested by Bismut [Bi] in his proof of the degenerate Morse inequalities of Bott. However our proof is rather different from [Bi] even in the case $[\omega] = 0$. We entirely avoid the probabilistic analysis of the heat kernels, which is the most difficult part of [Bi]. Instead, we give an explicit estimate on the number of the "small" eigenvalues of the deformed Laplacian. We now will explain briefly the main steps of the proof.

Let U be a small tubular neighborhood of C in M. We identify U with a neighborhood of the zero section in the normal bundle $\nu(C)$. Fix an affine connection on $\nu(C)$. This connection defines a bigrading

$$\Omega^\bullet(M, F) = \bigoplus \Omega^{i,j}(M, F),$$

where $\Omega^{i,j}(M, F)$ is the space of forms having degree i in the horizontal direction and degree j in the vertical direction. For $s \in \mathbb{R}$, let τ_s be the map from $\Omega^\bullet(M, F)$ to itself which sends $\alpha \in \Omega^{i,j}(M, F)$ to $s^j\alpha$.

Following Bismut, we introduce a 2-parameter deformation

$$\nabla_{t,\alpha} : \Omega^\bullet(M, F) \to \Omega^{\bullet+1}(M, F), \qquad t, \alpha \in \mathbb{R} \tag{0.7}$$

of the covariant derivative ∇, such that, for large values of t, α the Betti numbers of the deformed de Rham complex $\left(\Omega^\bullet(M, F), \nabla_{t,\alpha}\right)$ are equal to the Novikov numbers $\beta_p(\xi, F)$. Let $e(\omega) : \Omega^\bullet(M, F) \to \Omega^{\bullet+1}(M, F)$ denote the external multiplication by ω. Then on U the deformation (0.7) is given by

$$\nabla_{t,\alpha} = (\tau_{\sqrt{t}})^{-1} \circ \left(\nabla + t\alpha e(\omega)\right) \circ \tau_{\sqrt{t}}, \tag{0.8}$$

while outside of some larger neighborhood $V \supset U$, we have $\nabla_{t,\alpha} = \nabla + t\alpha e(\omega)$.

There exists a unique function $f : U \to \mathbb{R}$ such that $df = \omega$ and $f_{|C} = 0$. By the parameterized Morse lemma there exist an Euclidean metric $h^{\nu(C)}$ on $\nu(C)$ such that $\nu(C)$ decomposes into an orthogonal direct sum $\nu(C) = \nu^+(C) \oplus \nu^-(C)$ and if $(y^+, y^-) \in U$, then $f(y) = \frac{|y^+|^2}{2} - \frac{|y^-|^2}{2}$. Fix an arbitrary Riemannian metric g^C on C. The metrics $h^{\nu(C)}, g^C$ define naturally a Riemannian metric $g^{\nu(C)}$ on $\nu(C)$ (here we consider $\nu(C)$ as a non-compact manifold).

Let g^M be any Riemannian metric on M whose restriction to U is equal to $g^{\nu(C)}$. We also choose a Hermitian metric h^F on F. Let us denote by $\Delta_{t,\alpha}$ the Laplacian associated with the differential (0.7) and with the metrics g^M, h^F.

Fix $\alpha > 0$ sufficiently large. It turns out that, when $t \to \infty$, the eigenfunctions of $\Delta_{t,\alpha}$ corresponding to "small" eigenvalues localize near the critical points set C of ω. Hence, the number of the "small" eigenvalues of $\Delta_{t,\alpha}$ may be calculated by means of the restriction of $\Delta_{t,\alpha}$ on U. We are led, thus, to study of a certain Laplacian on $\nu(C)$. The latter Laplacian may be decomposed as $\bigoplus_Z \Delta^Z_{t,\alpha}$ where the sum ranges over all connected components of C and $\Delta^Z_{t,\alpha}$ is a Laplacian on the normal bundle $\nu(Z) = \nu(C)_{|Z}$ to Z. We denote by $\Delta^{Z,p}_{t,\alpha}$ $(p = 0, 1, 2, \dots)$ the restriction of $\Delta^Z_{t,\alpha}$ on the space of p-forms.

It follows from (0.8), that the spectrum of $\Delta^Z_{t,\alpha}$ does not depend on t. Moreover, if $\alpha > 0$ is sufficiently large, then

$$\dim \operatorname{Ker} \Delta^{Z,p}_{t,\alpha} = \dim H^{p-\operatorname{ind}(Z)}(Z, F_{|Z} \otimes o(Z)). \tag{0.9}$$

In the case when F is a trivial line bundle, (0.9) is proven by Bismut [Bi, Theorem 2.13].

Let $E^\bullet_{t,\alpha}$ $(p = 0, 1, \dots, n)$ be the subspace of $\Omega^\bullet(M, F)$ spanned by the eigenvectors of $\Delta_{t,\alpha}$ corresponding to the "small" eigenvalues. The cohomology of the deformed de Rham complex $\big(\Omega^\bullet(M, F), \nabla_{t,\alpha}\big)$ may be calculated as the cohomology of the subcomplex $\big(E^\bullet_{t,\alpha}, \nabla_{t,\alpha}\big)$.

Using the method of [Sh1], we show that, if the parameters t and α are large enough, then

$$\dim E^p_{t,\alpha} = \sum_Z \dim \operatorname{Ker} \Delta^{Z,p}_{t,\alpha}, \tag{0.10}$$

where the sum ranges over all connected components Z of C. The Theorem **??** follows now from (0.9),(0.10) by standard arguments (cf. [Bo2]).

9. L^2 **generalization.** Our Theorem **??**, combined with the results of W.Lück [Lü], gives the following L^2 version of the Novikov-Bott inequalities (5). Recall that L^2 generalization of the usual Morse inequalities for Morse functions were obtained first by S.P.Novikov and M.A.Shubin in [NS]. L^2-version of Novikov inequalities for 1-forms (allowing only isolated critial points) is considered in a recent preprint [MS] of V.Mathai and M.Shubin. They use different technique and their assumptions do not require resudual finiteness.

Let π be a countable residually finite group and let $N(\pi)$ denote the von Neumann algebra of π acting on the Hilbert space $l^2(\pi)$ from the left and commuting with the standard action of π on $l^2(\pi)$ from the right. The algebra $N(\pi)$ is supplied with the canonical finite trace and all von Neumann dimensions later will be understood with respect to this trace.

Suppose that a flat bundle L^π of Hilbert $N(\pi)$-modules $l^2(\pi)$ over a closed manifold M is given. (Here π is not necessarily the fundamental group of M). Any such bundle can be constructed by the standard construction from a representation of the fundamental group of M into π. Let F denote a finite dimensional flat vector bundle over M as above. The tensor product $L^\pi \otimes F$ (the tensor product taken over $\mathbb{C}$) is again a bundle of Hilbert $N(\pi)$-modules over M.

Let ω be a closed real valued 1-form on M, which is non-degenetrate in the sense of Bott. It determines a family of flat bundles F_t as in Section 2. Then there exists a countable subset $S \in \mathbb{R}$ (the set of *jump points*) such that the von Neumann dimension

$$\dim_{N(\pi)} H^i_{(2)}(M, L^\pi \otimes F_t)$$

is constant for $t \notin S$ and it jumps up for $t \in S$. This fact follows from Theorem 0.1 of Lück [Lü]. We will define *the von Neumann - Novikov numbers* $\beta_i(\xi, L^\pi \otimes F)$ as the value of $\dim_{N(\pi)} H^i_{(2)}(M, L^\pi \otimes F_t)$ for $t \notin S$. This value clearly depends only on the cohomology class $\xi \in H^1(M, \mathbb{R})$ of ω. Define *the von Neumann - Novikov polynomial*

$$N_{\xi, L^\pi \otimes F}(\lambda) \ = \ \sum \lambda^i \beta_i(\xi, L^\pi \otimes F).$$

For any component Z of the set of critical points C of ω define the following *von Neumann - Poincaré polynomial*

$$P_{Z, L^\pi \otimes F}(\lambda) \ = \ \sum \lambda^i \dim_{N(\pi)} H^i_{(2)}(Z, L^\pi_{|Z} \otimes F_{|Z} \otimes o(Z)),$$

and then the *von Neumann - Morse counting polynomial*

$$M_{\omega, L^\pi \otimes F}(\lambda) \ = \sum_Z \lambda^{\mathrm{ind}(Z)} P_{Z, L^\pi \otimes F}(\lambda),$$

the sum is taken over the set of connected components Z of C.

L2 There exists a polynomial $Q(\lambda)$ with real non-negative coefficients, such that

$$M_{\omega, L^\pi \otimes F}(\lambda) - N_{\xi, L^\pi \otimes F}(\lambda) = (1 + \lambda) Q(\lambda).$$

The proof is based on theorem (0.1) of Lück [Lü] and Theorem **??**. Let's briefly indicate the main points.

Let $\pi \supset \Gamma_1 \supset \Gamma_2 \supset \dots$ be a sequence of subgroups of finte index in π having trivial intersection. The flat $N(\pi)$ bundle L^π is constructed by means of a representation $\psi : \pi_1(M) \to \pi$. Let $\psi_m : \pi_1(M) \to \pi/\Gamma_m$ denote the composition of ψ with the reduction modulo Γ_m. The representation ψ_m determines a flat vector bundle L^π_m whose fiber is the group ring $\mathbb{C}[\pi/\Gamma_m]$ for any m. Slightly generalizing theorem (0.1) of Lück [Lü], we obtain that

$$\dim_{N(\pi)} H^i_{(2)}(M, L^\pi \otimes F) = \lim_{m \to \infty} |\pi/\Gamma_m|^{-1} \dim_{\mathbb{C}} H^i(M, L^\pi_m \otimes F)$$

This allows to approximate the von Neumann - Novikov polynomial $N_{\xi, L^\pi \otimes F}(\lambda)$ by the polynomials

$$|\pi/\Gamma_m|^{-1} N_{\xi, L^\pi_m \otimes F}(\lambda).$$

Similarly, the von Neumann - Morse polynomial $M_{\omega, L^\pi \otimes F}(\lambda)$ is approximated by the polynomials $|\pi/\Gamma_m|^{-1} M_{\omega, L^\pi_m \otimes F}(\lambda)$. Application of Theorem **??** then finishes the proof.

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