

INDEX THEOREM FOR EQUIVARIANT DIRAC OPERATORS ON NON-COMPACT MANIFOLDS

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ABSTRACT. Let D be a (generalized) Dirac operator on a non-compact complete Riemannian manifold M acted on by a compact Lie group G . Let $\mathbf{v} : M \rightarrow \mathfrak{g} = \text{Lie } G$ be an equivariant map, such that the corresponding vector field on M does not vanish outside of a compact subset. These data define an element of K -theory of the transversal cotangent bundle to M . Hence, by embedding of M into a compact manifold, one can define a topological index of the pair (D, \mathbf{v}) as an element of the completed ring of characters of G .

We define an analytic index of (D, \mathbf{v}) as an index space of certain deformation of D and we prove that the analytic and topological indexes coincide.

As a main step of the proof, we show that index is an invariant of a certain class of cobordisms, similar to the one considered by Ginzburg, Guillemin and Karshon. In particular, this means that the topological index of Atiyah is also invariant under this class of non-compact cobordisms.

As an application we extend the Atiyah-Segal-Singer equivariant index theorem to our non-compact setting. In particular, we obtain a new proof of this theorem for compact manifolds.

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1. INTRODUCTION

Suppose M is a complete Riemannian manifold, on which a compact Lie group G acts by isometries. To construct an index theory of Dirac-type operators on M , one needs some additional structure on M , which replaces the compactness. In this paper, this additional structure is a G -equivariant map $\mathbf{v} : M \rightarrow \mathfrak{g} = \text{Lie } G$, such that the induced vector field v on M does not vanish anywhere outside of a compact subset of M . We call \mathbf{v} a *taming* map, and we refer to the pair (M, \mathbf{v}) as a *tamed G -manifold*.

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a G -equivariant \mathbb{Z}_2 -graded self-adjoint Clifford module over M . We refer to the pair $(\mathcal{E}, \mathbf{v})$ as a *tamed Clifford module*.

The pair $(\mathcal{E}, \mathbf{v})$ defines an element in K -theory $K_G(T_G^*M)$ of transversal cotangent bundle, cf. [1] and Subsection 5.4 of this paper. Thus, using an embedding of (a compact part of M) into a closed manifold and the excision property (Th. 3.7 of [1]), one can define an index of $(\mathcal{E}, \mathbf{v})$ as an element of the completed ring of characters of G , cf. Subsection 5.1. We will refer to this index as the *topological index* of the tamed Clifford module $(\mathcal{E}, \mathbf{v})$ and we will denote it by $\chi_G^{\text{top}}(\mathcal{E}, \mathbf{v})$. This index was extensively studied by M. Vergne [20] and P.-E. Paradan [14, 15].

The goal of this paper is to construct an analytic counterpart of the topological index.

More precisely, we consider a Dirac operator $D^\pm : L^2(M, \mathcal{E}^\pm) \rightarrow L^2(M, \mathcal{E}^\mp)$ associated to a Clifford connection on \mathcal{E} (here $L^2(M, \mathcal{E})$ denotes the space of square-integrable sections of \mathcal{E}). Let $f : M \rightarrow [0, \infty)$ be a G -invariant function which increases fast enough at infinity (see Subsection 2.5 for the precise condition on f). We consider the *deformed Dirac operator* $D_{fv} = D + \sqrt{-1}c(fv)$, where $c : TM \simeq T^*M \rightarrow \text{End } \mathcal{E}$ is the Clifford module structure on \mathcal{E} . It turns out, cf. Theorem 2.9, that *each irreducible representation of G appears in $\text{Ker } D_{fv}$ with finite multiplicities*. In other words, the kernel of the deformed Dirac operator decomposes, as a Hilbert space, into (an infinite) direct sum

$$\text{Ker } D_{fv}^\pm = \sum_{V \in \text{Irr } G} m_V^\pm \cdot V. \quad (1.1)$$

Moreover, *the differences, $m_V^+ - m_V^-$ are independent of the choice of the function f and the Clifford connection, used in the definition of D* . Hence, these are invariants of the tamed Clifford module $(\mathcal{E}, \mathbf{v})$. We define the *analytic index* of $(\mathcal{E}, \mathbf{v})$ by the formula

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) := \sum_{V \in \text{Irr } G} (m_V^+ - m_V^-) \cdot V.$$

The main result of the paper is the *index theorem 5.5*, which states that the analytic and topological indexes coincide. The proof is based on an accurate study of the properties of the analytic index. Some of these properties will lead to new properties of the topological index via our index theorem. More generally, the index formula allows us to combine the analytic methods of this paper with the K -theoretical methods developed by P.-E. Paradan in [14, 15]. Some simple examples are presented below. For a more interesting application we refer the reader to [6].

In Section 3, we introduce the notion of cobordism between tamed Clifford modules. Roughly speaking, this is a usual cobordism, which carries a taming map. Our notion of cobordism is very close to the notion of non-compact cobordism developed by V. Ginzburg, V. Guillemin and Y. Karshon [10, 12, 9]. We prove, that *the index is preserved by a cobordism*. This result is the main technical tool in this paper.

Suppose $\Sigma \subset M$ is a compact G -invariant hypersurface, such that the vector field v does not vanish anywhere on Σ . We endow the open manifold $M \setminus \Sigma$ with a complete Riemannian metric and we denote by $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ the induced tamed Clifford module on $M \setminus \Sigma$. In Section 4, we prove that *the tamed Clifford modules $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ and $(\mathcal{E}, \mathbf{v})$ are cobordant. In particular, they have the same index*. We refer to this result as the *gluing formula*. Note, that the gluing formula is a generalization of the excision property for the index of transversally elliptic symbol, cf. Th. 3.7 of [1].

It is worth noting that the gluing formula gives a non-trivial new result even if M is compact. In this case, it expresses the usual equivariant index of \mathcal{E} in terms of the index of a Dirac operator on a non-compact, but, possibly, much simpler, manifold M_Σ .

The gluing formula takes especially nice form if Σ divides M into 2 disjoint manifolds M_1 and M_2 . Let $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ be the restrictions of $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ to M_1 and M_2 , respectively. Then the gluing formula implies

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) = \chi_G^{\text{an}}(\mathcal{E}_1, \mathbf{v}_1) + \chi_G^{\text{an}}(\mathcal{E}_2, \mathbf{v}_2).$$

In other words, *the index is additive*. This shows that the index theory of non-compact manifolds is, in a sense, simpler than that of compact manifolds (cf. [13], where a more complicated gluing formula for compact manifolds is obtained).

In Section 5, we use the gluing formula to prove that the topological and analytical indexes of tamed Clifford modules coincide. To this end we, first, consider a G -invariant open relatively compact set $U \subset M$ with smooth boundary which contains all the zeros of the vector field v . We endow U with a complete Riemannian metric and we denote by $(\mathcal{E}_U, \mathbf{v}_U)$ the induced tamed Clifford module over U . As an easy consequence of the gluing formula we obtain

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) = \chi_G^{\text{an}}(\mathcal{E}_U, \mathbf{v}_U).$$

We then embed U into a compact manifold N . By definition, cf. Subsection 5.4, the topological index $\chi_G^{\text{top}}(\mathcal{E}, \mathbf{v})$ is equal to the index of a certain transversally elliptic operator P on N . In Section 14 we give an explicit construction of such an operator and by direct computations show that its index is equal to $\chi_G^{\text{an}}(\mathcal{E}_U, \mathbf{v}_U)$. We, thus, obtain the *index formula*

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) = \chi_G^{\text{top}}(\mathcal{E}, \mathbf{v}). \tag{1.2}$$

Atiyah, [1], showed that the kernel of a transversally elliptic operator P is a *trace class* representation of G in the sense that $g \mapsto \text{Tr}(g|_{\text{Ker } P})$, $g \in G$ is well defined as a distribution on G . It follows now from the index formula (1.2) that the index space of the operator $D_{f_v} = D + fc(v)$

is a (virtual) representation of trace class. In other words the sum

$$T(g) = \sum_{V \in \text{Irr } G} (m_V^+ - m_V^-) \text{Tr}(g|_V)$$

converges to a distribution on G (here m_V^\pm are as in (1.1)). We don't know any direct analytic proof of this fact. In particular, we don't know whether the individual sums $\sum_{V \in \text{Irr } G} m_V^\pm \text{Tr}(g|_V)$ converge to distributions on G .

As another application of the index formula, we see that the topological index of Atiyah is invariant under our non-compact cobordism. In particular, it satisfies the gluing formula. This may be viewed as a generalization of the excision theorem 3.7 of [1].

In Section 7, we consider the case when G is a torus. Let $F \subset M$ be the set of points fixed by the action of G . Assume that the vector field v does not vanish anywhere outside of F . In Subsection 7.1, we show that $(\mathcal{E}, \mathbf{v})$ is cobordant to a Clifford module over the normal bundle to F . This leads to an extension of the Atiyah-Segal-Singer equivariant index theorem to our non-compact setting. As a byproduct, we obtain a new proof of the classical Atiyah-Segal-Singer theorem. This proof is an analytic analogue of the proof given by Atiyah [1, Lect. 6], Vergne [20, Part II] and Paradan [14, §4].

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2. INDEX ON NON-COMPACT MANIFOLDS

In this section we introduce our main objects of study: tamed non-compact manifolds, tamed Clifford modules, and the (analytic) equivariant index of such modules.

2.1. Clifford module and Dirac operator. First, we recall the basic properties of Clifford modules and Dirac operators. When possible, we follow the notation of [4].

Suppose (M, g^M) is a complete Riemannian manifold. Let $C(M)$ denote the Clifford bundle of M (cf. [4, §3.3]), i.e., a vector bundle, whose fiber at every point $x \in M$ is isomorphic to the Clifford algebra $C(T_x^*M)$ of the cotangent space.

Suppose $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a \mathbb{Z}_2 -graded complex vector bundle on M endowed with a graded action

$$(a, s) \mapsto c(a)s, \quad \text{where } a \in \Gamma(M, C(M)), s \in \Gamma(M, \mathcal{E}),$$

of the bundle $C(M)$. We say that \mathcal{E} is a (\mathbb{Z}_2 -graded self-adjoint) *Clifford module* on M if it is equipped with a Hermitian metric such that the operator $c(v) : \mathcal{E}_x \rightarrow \mathcal{E}_x$ is skew-adjoint, for all $x \in M$ and $v \in T_x^*M$.

A *Clifford connection* on \mathcal{E} is a Hermitian connection $\nabla^\mathcal{E}$, which preserves the subbundles \mathcal{E}^\pm and

$$[\nabla_X^\mathcal{E}, c(a)] = c(\nabla_X^{\text{LC}} a), \quad \text{for any } a \in \Gamma(M, C(M)), X \in \Gamma(M, TM),$$

where ∇_X^{LC} is the Levi-Civita covariant derivative on $C(M)$ associated with the Riemannian metric on M .

The *Dirac operator* $D : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, \mathcal{E})$ associated to a Clifford connection $\nabla^\mathcal{E}$ is defined by the following composition

$$\Gamma(M, \mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Gamma(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} \Gamma(M, \mathcal{E}).$$

In local coordinates, this operator may be written as $D = \sum c(dx^i) \nabla_{\partial_i}^\mathcal{E}$. Note that D sends even sections to odd sections and vice versa: $D : \Gamma(M, \mathcal{E}^\pm) \rightarrow \Gamma(M, \mathcal{E}^\mp)$.

Consider the L^2 -scalar product on the space of sections $\Gamma(M, \mathcal{E})$ defined by the Riemannian metric on M and the Hermitian structure on \mathcal{E} . By [4, Proposition 3.44], the Dirac operator associated to a Clifford connection $\nabla^\mathcal{E}$ is formally self-adjoint with respect to this scalar product. Moreover, it is essentially self-adjoint with the initial domain smooth, compactly supported sections, cf. [7], [11, Th. 1.17].

If W is a manifold with boundary, then by a Clifford module over it we will understand a smooth vector bundle \mathcal{E} over W , whose restriction to the set of the interior points W^{int} of W has a structure of a Clifford module over W^{int} . (Usually we will require some additional structure of \mathcal{E} near the boundary of W , but we will formulate these requirements when we need them.)

2.2. Group action. The index. Suppose that a compact Lie group G acts on M by isometries. Assume that there is given a lift of this action to \mathcal{E} , which preserves the grading, the connection and the Hermitian metric on \mathcal{E} . Then the Dirac operator D commutes with the action of G . Hence, $\text{Ker } D$ is a G -invariant subspace of the space $L^2(M, \mathcal{E})$ of square-integrable sections of \mathcal{E} .

If M is compact, then $\text{Ker } D^\pm$ is finite dimensional. Hence, it breaks into a finite sum $\text{Ker } D^\pm = \sum_{V \in \text{Irr } G} m_V^\pm V$, where the sum is taken over the set $\text{Irr } G$ of all irreducible representations of G . This allows one to defined the *index*

$$\chi_G(D) = \sum_{V \in \text{Irr } G} (m_V^+ - m_V^-) \cdot V, \quad (2.1)$$

as a virtual representation of G .

Unlike the numbers m_V^\pm , the differences $m_V^+ - m_V^-$ do not depend on the choice of the connection $\nabla^\mathcal{E}$ and the metric $h^\mathcal{E}$. Hence, the index $\chi_G(D)$ depends only on M and the equivariant Clifford module $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. We set $\chi_G(\mathcal{E}) := \chi_G(D)$, and refer to it as the *index* of \mathcal{E} .

2.3. A tamed non-compact manifold. The main purpose of this paper is to define and study an analogue of (2.1) for a G -equivariant Clifford module over a complete *non-compact* manifold. For this we need and additional structure on M . This structure is given by an equivariant map

$\mathbf{v} : M \rightarrow \mathfrak{g}$, where \mathfrak{g} denotes the Lie algebra of G and G acts on it by the adjoint representation. Note that such a map induces a vector field v on M defined by

$$v(x) := \left. \frac{d}{dt} \right|_{t=0} \exp(t\mathbf{v}(x)) \cdot x. \quad (2.2)$$

In the sequel, we will always denote maps to \mathfrak{g} by bold letters and the vector fields on M induced by these maps by ordinary letters.

Definition 2.4. *Let M be a complete G -manifold. A taming map is a G -equivariant map $\mathbf{v} : M \rightarrow \mathfrak{g}$, such that the vector field v on M , defined by (2.2), does not vanish anywhere outside of a compact subset of M . If \mathbf{v} is a taming map, we refer to the pair (M, \mathbf{v}) as a tamed G -manifold.*

If, in addition, \mathcal{E} is a G -equivariant \mathbb{Z}_2 -graded self-adjoint Clifford module over M , we refer to the pair $(\mathcal{E}, \mathbf{v})$ as a tamed Clifford module over M .

The index we are going to define depends on the (equivalence class) of \mathbf{v} .

2.5. A rescaling of v . Our definition of the index uses certain rescaling of the vector field v . By this we mean the product $f(x)v(x)$, where $f : M \rightarrow [0, \infty)$ is a smooth positive function. Roughly speaking, we demand that $f(x)v(x)$ tends to infinity “fast enough” when x tends to infinity. The precise conditions we impose on f are quite technical, cf. Definition 2.6. Luckily, our index turns out to be independent of the concrete choice of f . It is important, however, to know that at least one admissible function exists. This is guaranteed by Lemma 2.7 below.

We need to introduce some additional notations.

For a vector $\mathbf{u} \in \mathfrak{g}$, we denote by $\mathcal{L}_{\mathbf{u}}^{\mathcal{E}}$ the infinitesimal action of \mathbf{u} on $\Gamma(M, \mathcal{E})$ induced by the action on G on \mathcal{E} . On the other side, we can consider the covariant derivative $\nabla_{\mathbf{u}}^{\mathcal{E}} : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, \mathcal{E})$ along the vector field u induced by \mathbf{u} . The difference between those two operators is a bundle map, which we denote by

$$\mu^{\mathcal{E}}(\mathbf{u}) := \nabla_{\mathbf{u}}^{\mathcal{E}} - \mathcal{L}_{\mathbf{u}}^{\mathcal{E}} \in \text{End } \mathcal{E}. \quad (2.3)$$

We will use the same notation $|\cdot|$ for the norms on the bundles TM, T^*M, \mathcal{E} . Let $\text{End}(TM)$ and $\text{End}(\mathcal{E})$ denote the bundles of endomorphisms of TM and \mathcal{E} , respectively. We will denote by $\|\cdot\|$ the norms on these bundles induced by $|\cdot|$. To simplify the notation, set

$$\nu = |\mathbf{v}| + \|\nabla^{\text{LC}} v\| + \|\mu^{\mathcal{E}}(\mathbf{v})\| + |v| + 1. \quad (2.4)$$

Definition 2.6. *We say that a smooth G -invariant function $f : M \rightarrow [0, \infty)$ on a tamed G -manifold (M, \mathbf{v}) is admissible for the triple $(\mathcal{E}, \mathbf{v}, \nabla^{\mathcal{E}})$ if*

$$\lim_{M \ni x \rightarrow \infty} \frac{f^2 |v|^2}{|df||v| + f\nu + 1} = \infty. \quad (2.5)$$

Lemma 2.7. *Let $(\mathcal{E}, \mathbf{v})$ be a tamed Clifford module and let $\nabla^{\mathcal{E}}$ be a G -invariant Clifford connection on \mathcal{E} . Then there exists an admissible function f for the triple $(\mathcal{E}, \mathbf{v}, \nabla^{\mathcal{E}})$.*

We prove the lemma in Section 8 as a particular case of a more general Lemma 8.3.

2.8. Index on non-compact manifolds. We use the Riemannian metric on M , to identify the tangent and the cotangent bundles to M . In particular, we consider v as a section of T^*M .

Let f be an admissible function. Consider the *deformed Dirac operator*

$$D_{fv} = D + \sqrt{-1}c(fv). \quad (2.6)$$

This is again a G -invariant essentially self-adjoint operator on M , cf. the remark on page 411 of [7].

Our first result is the following

Theorem 2.9. *Suppose f is an admissible function. Then*

1. *The kernel of the deformed Dirac operator D_{fv} decomposes, as a Hilbert space, into an infinite direct sum*

$$\text{Ker } D_{fv}^\pm = \sum_{V \in \text{Irr } G} m_V^\pm \cdot V. \quad (2.7)$$

In other words, each irreducible representation of G appears in $\text{Ker } D_{fv}^\pm$ with finite multiplicity.

2. *The differences $m_V^+ - m_V^-$ ($V \in \text{Irr } G$) are independent of the choices of the admissible function f and the G -invariant Clifford connection on \mathcal{E} , used in the definition of D .*

The proof of the first part of the theorem is given in Section 9. The second part of the theorem will be obtained in Subsection 3.9 as an immediate consequence of Theorem 3.7 about cobordism invariance of the index.

We will refer to the pair (D, \mathbf{v}) as a *tamed Dirac operator*. The above theorem allows to defined the index of a tamed Dirac operator:

$$\chi_G(D, \mathbf{v}) := \chi_G(D_{fv})$$

using (2.1). Note, however, that now the sum in the right hand side of (2.1) is infinite.

Since $\chi_G(D, \mathbf{v})$ is independent of the choice of the connection on \mathcal{E} , it is an invariant of the tamed Clifford module $(\mathcal{E}, \mathbf{v})$. This allows us to define the *(analytic) index of a tamed Clifford module* $(\mathcal{E}, \mathbf{v})$ by $\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) := \chi_G(D, \mathbf{v})$, where D is the Dirac operator associated to some G -invariant Clifford connection on \mathcal{E} .

Most of this paper is devoted to the study of the properties of $\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v})$. In Section 3 we will show that it is invariant under certain class of cobordisms. In particular, this implies that $\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v})$ depends only on the cobordism class of the map \mathbf{v} . In some cases, one can give a very simple topological description of the cobordism classes of \mathbf{v} . In the next subsection, we do it for the most important for applications case of *topologically tame manifolds*.

2.10. Topologically tame manifolds. Recall that a (non-compact) manifold M is called *topologically tame* if it is diffeomorphic to the interior of a compact manifold \bar{M} with boundary.

Suppose M is a topologically tame manifold and let us fix a diffeomorphism ϕ between M and the interior of a compact manifold with boundary \bar{M} . A small neighborhood U of the boundary $\partial\bar{M}$ of \bar{M} can be identified as

$$U \simeq \partial\bar{M} \times [0, 1). \quad (2.8)$$

Let $\mathbf{v} : M \rightarrow \mathfrak{g}$ be a taming map. Then it induces, via ϕ and (2.8), a map $\partial\bar{M} \times [0, 1) \rightarrow \mathfrak{g}$, which we will also denote by \mathbf{v} . Hence, for each $t \in [0, 1)$, we have a map $\mathbf{v}_t : \partial\bar{M} \rightarrow \mathfrak{g}$, obtained by restricting \mathbf{v} to $\partial\bar{M} \times \{t\}$. It follows from Definition 2.4, that $\mathbf{v}_t(x) \neq 0$ for small t and all $x \in \partial\bar{M}$. Thus $\mathbf{v}_t(x)/\|\mathbf{v}_t(x)\|$ defines a map from $\partial\bar{M}$ to the unit sphere $S_{\mathfrak{g}}$ in \mathfrak{g} . Clearly, the homotopy class of this map does not depend on t , nor on the choice of the splitting (2.8). We denote by $\sigma(\mathbf{v})$ the obtained homotopy class of maps $\partial\bar{M} \rightarrow S_{\mathfrak{g}}$. The following proposition is a direct consequence of cobordism invariance of the index (Theorem 3.7).

Proposition 2.11. *If M is a topologically tame manifold, then the index $\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v})$ does not change if we change the map $\mathbf{v} : M \rightarrow \mathfrak{g}$, provided $\sigma(\mathbf{v})$ does not change.*

3. COBORDISM INVARIANCE OF THE INDEX

In this section we introduce the notion of cobordism between tamed Clifford modules and tamed Dirac operators. We show that the analytic index, introduced in Subsection 2.8, is invariant under a cobordism. This result will serve as a main technical tool throughout the paper. In particular, we use it in the end of this section to show that the index is independent of the choice of the admissible function and the Clifford connection on \mathcal{E} .

3.1. Cobordism between tamed G -manifolds. Note, first, that, for cobordism to be meaningful, one must make some compactness assumption. Otherwise, every manifold is cobordant to the empty set via the noncompact cobordism $M \times [0, 1)$. Since our manifolds are non-compact themselves, we can not demand cobordism to be compact. Instead, we demand the cobordism to carry a taming map to \mathfrak{g} .

Definition 3.2. *A cobordism between tamed G -manifolds (M_1, \mathbf{v}_1) and (M_2, \mathbf{v}_2) is a triple (W, \mathbf{v}, ϕ) , where*

- (i) W is a complete Riemannian G -manifold with boundary;
- (ii) $\mathbf{v} : W \rightarrow \mathfrak{g}$ is a smooth G -invariant map, such that the corresponding vector field v does not vanish anywhere outside of a compact subset of W ;
- (iii) ϕ is a G -equivariant, metric preserving diffeomorphism between a neighborhood U of the boundary ∂W of W and the disjoint union $(M_1 \times [0, \varepsilon)) \sqcup (M_2 \times (-\varepsilon, 0])$. We will refer to U as the neck and we will identify it with $(M_1 \times [0, \varepsilon)) \sqcup (M_2 \times (-\varepsilon, 0])$.
- (iv) the restriction of $\mathbf{v}(\phi^{-1}(x, t))$ to $M_1 \times [0, \varepsilon)$ (resp. to $M_2 \times (-\varepsilon, 0])$ is equal to $\mathbf{v}_1(x)$ (resp. to $\mathbf{v}_2(x)$).

Remark 3.3. A cobordism in the sense of Definition 3.2 is also a cobordism in the sense of Guillemin, Ginzburg and Karshon [10, 12, 9]. If G is a circle, one can take $|fv|^2$ (where f is an admissible function) as an abstract moment map. It is not difficult to construct an abstract moment map out of \mathbf{v} in the general case.

3.4. Cobordism between tamed Clifford modules. We now discuss our main notion – the cobordism between tamed Clifford modules and tamed Dirac operators. Before giving the precise definition let us fix some notation.

If M is a Riemannian G -manifold, then, for any interval $I \subset \mathbb{R}$, the product $M \times I$ carries natural Riemannian metric and G -action. Let $\pi : M \times I \rightarrow M$, $t : M \times I \rightarrow I$ denote the natural projections. We refer to the pull-back $\pi^*\mathcal{E}$ as a vector bundle *induced* by \mathcal{E} . We view t as a real valued function on M , and we denote by dt its differential.

Definition 3.5. Let (M_1, \mathbf{v}_1) and (M_2, \mathbf{v}_2) be tamed G -manifolds. Suppose that each M_i , $i = 1, 2$, is endowed with a G -equivariant self-adjoint Clifford module $\mathcal{E}_i = \mathcal{E}_i^+ \oplus \mathcal{E}_i^-$. A cobordism between the tamed Clifford modules $(\mathcal{E}_i, \mathbf{v}_i)$, $i = 1, 2$, is a cobordism (W, \mathbf{v}, ϕ) between (M_i, \mathbf{v}_i) together with a pair (\mathcal{E}_W, ψ) , where

- (i) \mathcal{E}_W is a G -equivariant (non-graded) self-adjoint Clifford module over W ;
- (ii) ψ is a G -equivariant isometric isomorphism between the restriction of \mathcal{E}_W to U and the Clifford module induced on the neck $(M_1 \times [0, \varepsilon)) \sqcup (M_2 \times (-\varepsilon, 0])$ by \mathcal{E}_i .
- (iii) On the neck U we have $c(dt)|_{\psi^{-1}\mathcal{E}_i^\pm} = \pm\sqrt{-1}$.

In the situation of Definition 3.5, we say that the tamed Clifford modules $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ are *cobordant* and we refer to $(\mathcal{E}_W, \mathbf{v})$ as a cobordism between these modules.

Remark 3.6. Let $\mathcal{E}_1^{\text{op}}$ denote the Clifford module \mathcal{E}_1 with the opposite grading, i.e., $\mathcal{E}_1^{\text{op}\pm} = \mathcal{E}_1^\mp$. Then, $\chi_G^{\text{an}}(\mathcal{E}_1, \mathbf{v}_1) = -\chi_G^{\text{an}}(\mathcal{E}_1^{\text{op}}, \mathbf{v}_1)$.

Consider the Clifford module \mathcal{E} over the disjoint union $M = M_1 \sqcup M_2$ induced by the Clifford modules $\mathcal{E}_1^{\text{op}}$ and \mathcal{E}_2 . Let $\mathbf{v} : M \rightarrow \mathfrak{g}$ be the map such that $\mathbf{v}|_{M_i} = \mathbf{v}_i$. A cobordism between $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ may be viewed as a cobordism between $(\mathcal{E}, \mathbf{v})$ and (the Clifford module over) the empty set.

One of the main results of this paper is the following theorem, which asserts that the index is preserved by a cobordism.

Theorem 3.7. Suppose $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ are cobordant tamed Clifford modules. Let D_1, D_2 be Dirac operators associated to G -invariant Clifford connections on \mathcal{E}_1 and \mathcal{E}_2 , respectively. Then, for any admissible functions f_1, f_2 ,

$$\chi_G(D_1 + \sqrt{-1}c(f_1v_1)) = \chi_G(D_2 + \sqrt{-1}c(f_2v_2)).$$

The proof of the theorem is given in Section 10. Here we only explain the main ideas of the proof.

3.8. The scheme of the proof. By Remark 3.6, it is enough to show that, if $(\mathcal{E}, \mathbf{v})$ is cobordant to (the Clifford module over) the empty set, then $\chi_G(D_{fv}) = 0$ for any admissible function f .

Let $(W, \mathcal{E}_W, \mathbf{v})$ be a cobordism between the empty set and $(\mathcal{E}, \mathbf{v})$ (slightly abusing the notation, we denote by the same letter \mathbf{v} the taming maps on W and M).

In Section 8 we define the notion of an admissible function on a cobordism $(W, \mathcal{E}_W, \mathbf{v})$ analogous to Definition 2.6. Moreover, we show (cf. Lemma 8.3) that, if f is an admissible functions on $(M, \mathcal{E}, \mathbf{v})$, then there exists an admissible function on $(W, \mathcal{E}_W, \mathbf{v})$, whose restriction to M equals f . By a slight abuse of notation, we will denote this function by the same letter f .

Let \tilde{W} be the manifold obtained from W by attaching a cylinder to the boundary, i.e.,

$$\tilde{W} = W \sqcup (M \times (0, \infty)).$$

The action of G , the Riemannian metric, the map \mathbf{v} , the function f and the Clifford bundle \mathcal{E}_W extend naturally from W to \tilde{W} .

Consider the exterior algebra $\Lambda^\bullet \mathbb{C} = \Lambda^0 \mathbb{C} \oplus \Lambda^1 \mathbb{C}$. It has two (anti)-commuting actions c_L and c_R (left and right action) of the Clifford algebra of \mathbb{R} , cf. Subsection 10.1. Define a grading of $\tilde{\mathcal{E}} = \mathcal{E} \otimes \Lambda^\bullet \mathbb{C}$ and a Clifford action $\tilde{c} : T^* \tilde{W} \rightarrow \text{End } \tilde{\mathcal{E}}$ by the formulas

$$\tilde{\mathcal{E}}^+ := \mathcal{E}_W \otimes \Lambda^0 \mathbb{C}; \quad \tilde{\mathcal{E}}^- := \mathcal{E}_W \otimes \Lambda^1 \mathbb{C}; \quad \tilde{c}(v) := \sqrt{-1} c(v) \otimes c_L(1) \quad (v \in T^* \tilde{W}).$$

Let \tilde{D} be a Dirac operator on $\tilde{\mathcal{E}}$ and consider the operator ${}^1 \tilde{D}_{fv} := \tilde{D} + \sqrt{-1} \tilde{c}(fv)$.

Let $p : \tilde{W} \rightarrow \mathbb{R}$ be a map, whose restriction to $M \times (1, \infty)$ is the projection on the second factor, and such that $p(W) = 0$. For every $a \in \mathbb{R}$, consider the operator $\mathbf{D}_a := \tilde{D}_{fv} - c_R((p(t) - a))$. Here, to simplify the notation, we write simply $c_R(\cdot)$ for the operator $1 \otimes c_R(\cdot)$. Then (cf. Lemma 10.4)

$$\mathbf{D}_a^2 = \tilde{D}_{fv}^2 - B + |p(x) - a|^2, \quad (3.1)$$

where $B : \Gamma(\tilde{W}, \tilde{\mathcal{E}}) \rightarrow \Gamma(\tilde{W}, \tilde{\mathcal{E}})$ is a bounded operator ².

It follows easily from (3.1) that the index $\chi_G(\mathbf{D}_a)$ is well defined and is independent of a , cf. Subsection 10.6. Moreover, $\chi_G(\mathbf{D}_a) = 0$ for $a \ll 0$ and, if $a > 0$ is very large, then all the sections in $\text{Ker } \mathbf{D}_a^2$ are concentrated on the cylinder $M \times (0, \infty)$, not far from $M \times \{a\}$ (this part of the proof essentially repeats the arguments of Witten in [21]). Hence, the calculation of $\text{Ker } \mathbf{D}_a^2$ is reduced to a problem on the cylinder $M \times (0, \infty)$. It is not difficult now to show that $\chi_G(\mathbf{D}_a) = \chi_G(D_{fv})$ for $a \gg 0$, cf. Theorem 10.8.

Theorem 3.7 follows now from the fact that $\chi_G(\mathbf{D}_a)$ is independent of a .

3.9. The definition of the analytic index of a tamed Clifford module. Theorem 3.7 implies, in particular, that, if $(\mathcal{E}, \mathbf{v})$ is a tamed Clifford module, then the index $\chi_G(D_{fv})$ is independent of the choice of the admissible function f and the Clifford connection on \mathcal{E} . This proves part 2 of Theorem 2.9 and (cf. Subsection 2.8) allows us to define the (*analytic*) *index* of the tamed Clifford module $(\mathcal{E}, \mathbf{v})$

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) := \chi_G(D_{fv}), \quad f \text{ is an admissible function.}$$

Theorem 3.7 can be reformulated now as

Theorem 3.10. *The analytic indexes of cobordant tamed Clifford modules coincide.*

¹Note that v might vanish somewhere near infinity on the cylindrical end of \tilde{W} . In particular, the index of \tilde{D}_{fv} is not defined in general.

²The reason that, contrary to (9.2), no covariant derivatives occur in (3.1) is that we used the right Clifford multiplication c_R to define the deformed Dirac operator \mathbf{D}_a . The crucial here is the fact, that c_R commutes with the left Clifford multiplication c_L , used in the definition of the Clifford structure on $\tilde{\mathcal{E}}$.

3.11. **Index and zeros of v .** As a simple corollary of Theorem 3.7, we obtain the following

Lemma 3.12. *If the vector field $v(x) \neq 0$ for all $x \in M$, then $\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) = 0$.*

Proof. Consider the product $W = M \times [0, \infty)$ and define the map $\tilde{\mathbf{v}} : W \rightarrow \mathfrak{g}$ by the formula: $\tilde{\mathbf{v}}(x, t) = \mathbf{v}(x)$. Clearly, $(W, \tilde{\mathbf{v}})$ is a cobordism between the tamed G -manifold M and the empty set. Let \mathcal{E}_W be the lift of \mathcal{E} to W . Define the Clifford module structure $c : T^*W \rightarrow \text{End } \mathcal{E}_W$ by the formula

$$c(x, a)e = c(x)e \pm \sqrt{-1}ae, \quad (x, a) \in T^*W \simeq T^*M \oplus \mathbb{R}, \quad e \in \mathcal{E}_W^\pm.$$

Then $(\mathcal{E}_W, \tilde{\mathbf{v}})$ is a cobordism between $(\mathcal{E}, \mathbf{v})$ and the Clifford module over the empty set. \square

3.13. **The stability of the index.** We will now amplify the above lemma and show that the index is independent of the restriction of $(\mathcal{E}, \mathbf{v})$ to a subset, where $v \neq 0$.

Let (M_i, \mathbf{v}_i) $i = 1, 2$, be tamed n -dimensional G -manifolds. Let U be an open n -dimensional G -manifold. For each $i = 1, 2$, let $\phi_i : U \rightarrow M_i$ be a smooth G -equivariant embedding. Set $U_i = \phi_i(U) \subset M_i$. Assume that the boundary $\Sigma_i = \partial U_i$ of U_i is a smooth hypersurfaces in M_i . Assume also that the vector field v_i induced by \mathbf{v}_i on M_i does not vanish anywhere on $M_i \setminus U_i$.

Lemma 3.14. *Let $(\mathcal{E}_1, \mathbf{v}_1)$, $(\mathcal{E}_2, \mathbf{v}_2)$ be tamed Clifford modules over M_1 and M_2 , respectively. Suppose that the pull-backs $\phi_i^* \mathcal{E}_i$, $i = 1, 2$ are G -equivariantly isomorphic as \mathbb{Z}_2 -graded self-adjoint Clifford modules over U . Assume also that $\mathbf{v}_1 \circ \phi_1 \equiv \mathbf{v}_2 \circ \phi_2$. Then $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$ are cobordant. In particular, $\chi_G^{\text{an}}(\mathcal{E}_1, \mathbf{v}_1) = \chi_G^{\text{an}}(\mathcal{E}_2, \mathbf{v}_2)$.*

The lemma is proven ³ in Section 12 by constructing an explicit cobordism between $(\mathcal{E}_1, \mathbf{v}_1)$ and $(\mathcal{E}_2, \mathbf{v}_2)$.

Remark 3.15. Lemma 3.14 implies that the index depends only on the information near the zeros of v . In particular, if G is a torus and $\mathbf{v} : M \rightarrow \mathfrak{g}$ is a constant map to a generic vector of \mathfrak{g} , this implies that the index is completely defined by the data near the fixed points of the action. This is, essentially, the equivariant index theorem of Atiyah-Segal-Singer (or, rather, its extension to non-compact manifolds). See Section 7 for more details.

The following lemma is, in a sense, opposite to Lemma 3.14. The combination of these 2 lemmas might lead to an essential simplification of a problem.

Lemma 3.16. *Let $\mathbf{v}_1, \mathbf{v}_2 : M \rightarrow \mathfrak{g}$ be taming maps, which coincide out of a compact subset of M . Then the tamed Clifford modules $(\mathcal{E}, \mathbf{v}_1)$ and $(\mathcal{E}, \mathbf{v}_2)$ are cobordant. In particular, $\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}_1) = \chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}_2)$.*

Proof. Consider the product $W = M \times [0, 1]$. Let $s : [0, 1] \rightarrow [0, 1]$ be a smooth increasing function, such that $s(t) = 0$ for $t \leq 1/3$ and $s(t) = 1$ for $t \geq 2/3$. Define the map $\tilde{\mathbf{v}} : W \rightarrow \mathfrak{g}$ by

³One can note that Lemma 3.14 follows immediately from Lemma 3.12 and the additivity of the index stated in Corollary 4.7. However, the fact that the tamed Clifford modules $\mathcal{E}_1, \mathcal{E}_2$ of Corollary 4.7 are well defined relies on Lemma 3.14. The lemma is also used in the proof of the additivity formula, cf. Section 13.

the formula $\tilde{\mathbf{v}}(x, t) = (1 - s(t))\mathbf{v}_1(x) + s(t)\mathbf{v}_2(x)$. Then $(W, \tilde{\mathbf{v}})$ is a cobordism between (M, \mathbf{v}_1) and (M, \mathbf{v}_2) . Let \mathcal{E}_W be the lift of \mathcal{E} to W , endowed with the Clifford module structure defined in the proof of Lemma 3.12. Then $(\mathcal{E}_W, \tilde{\mathbf{v}})$ is a cobordism between $(\mathcal{E}, \mathbf{v}_1)$ and $(\mathcal{E}, \mathbf{v}_2)$. \square

4. THE GLUING FORMULA

If we cut a tamed G -manifold along a G -invariant hypersurface Σ , we obtain a manifold with boundary. By rescaling the metric near the boundary we may convert it to a complete manifold without boundary, in fact, to a tamed G -manifold. In this section, we show that the index is invariant under this type of surgery. In particular, if Σ divides M into two pieces M_1 and M_2 , we see that the index on M is equal to the sum of the indexes on M_1 and M_2 . In other words, the index is *additive*. This property can be used for calculating the index on a *compact manifold* M (note that the manifolds M_1, M_2 are non-compact even if M is compact).

4.1. The surgery. Let (M, \mathbf{v}) be a tamed G -manifold. Suppose $\Sigma \subset M$ is a smooth G -invariant hypersurface in M . For simplicity, we assume that Σ is compact. Assume also that the vector field v induced by \mathbf{v} does not vanish anywhere on Σ . Suppose that $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a G -equivariant \mathbb{Z}_2 -graded self-adjoint Clifford module over M . Denote by \mathcal{E}_Σ the restriction of the \mathbb{Z}_2 -graded Hermitian vector bundle \mathcal{E} to $M_\Sigma := M \setminus \Sigma$.

Let g^M denote the Riemannian metric on M . By rescaling of g^M near Σ , one can obtain a complete Riemannian metric on $M_\Sigma := M \setminus \Sigma$, which makes $(M_\Sigma, \mathbf{v}_\Sigma := \mathbf{v}|_{M_\Sigma})$ a tamed G -manifold. It follows, from the cobordism invariance of the index (more precisely, from Lemma 3.14), that the concrete choice of this metric is irrelevant for our index theory. We, however, must show that one can choose such a metric and a Clifford module structure on \mathcal{E}_Σ consistently. This is done in the next subsection.

4.2. Choice of a metric on M_Σ and a Clifford module structure on \mathcal{E}_Σ . Let $\tau : M \rightarrow \mathbb{R}$ be a smooth G -invariant function, such that $\tau^{-1}(0) = \Sigma$ and there are no critical values of τ in the interval $[-1, 1]$. Let $r : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function, such that $r(t) = t^2$ for $|t| \leq 1/3$, $r(t) > 1/9$ for $|t| > 1/3$ and $r(t) \equiv 1$ for $|t| > 2/3$. Set $\alpha(x) = r(\tau(x))$. Define the metric g^{M_Σ} on M_Σ by the formula

$$g^{M_\Sigma} := \frac{1}{\alpha(x)^2} g^M. \quad (4.1)$$

This is a complete G -invariant metric on M_Σ . Hence, $(M_\Sigma, g^{M_\Sigma}, \mathbf{v}_\Sigma)$ is a tamed G -manifold.

Define a map $c_\Sigma : T^*M_\Sigma \rightarrow \text{End } \mathcal{E}_\Sigma$ by the formula

$$c_\Sigma := \alpha(x)c, \quad (4.2)$$

where $c : T^*M \rightarrow \text{End } \mathcal{E}$ is the Clifford module structure on \mathcal{E} . Then \mathcal{E}_Σ becomes a G -equivariant \mathbb{Z}_2 -graded self-adjoint Clifford module over M_Σ . The pair $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ is a tamed Clifford module.

4.3. Before formulating the theorem, let us make the following remark. Suppose we choose another complete G -invariant metric on M_Σ and another Clifford module structure on \mathcal{E}_Σ , which coincides with the ones chosen above on $\alpha^{-1}(1) \subset M$. Then, by Lemma 3.14, the obtained tamed Clifford module is cobordant to $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$. In view of this remark, we don't demand anymore that the Clifford structure on \mathcal{E}_Σ is given by (4.2). Instead, we fix a structure of a G -equivariant self-adjoint Clifford module on the bundle \mathcal{E}_Σ , such that $\mathcal{E}_\Sigma|_{\alpha^{-1}(1)} = \mathcal{E}|_{\alpha^{-1}(1)}$ and the corresponding Riemannian metric on M_Σ is complete.

Theorem 4.4. *The tamed Clifford modules $(\mathcal{E}, \mathbf{v})$ and $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$ are cobordant. In particular,*

$$\chi_G^{an}(\mathcal{E}, \mathbf{v}) = \chi_G^{an}(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma).$$

We refer to Theorem 4.4 as a *gluing formula*, meaning that M is obtained from M_Σ by gluing along Σ .

The proof of Theorem 4.4 is given in Section 13. Here we only present the main idea of how to construct the cobordism W between M and M_Σ .

4.5. **The idea of the proof of the gluing formula.** Consider the product $M \times [0, 1]$, and the set

$$Z := \{ (x, t) \in M \times [0, 1] : t \leq 1/3, x \in \Sigma \}.$$

Set $W := (M \times [0, 1]) \setminus Z$. Then W is a G -manifold, whose boundary is diffeomorphic to the disjoint union of $M \setminus \Sigma \simeq (M \setminus \Sigma) \times \{0\}$ and $M \simeq M \times \{1\}$. Essentially, W is the required cobordism. However, we have to be accurate in defining a complete Riemannian metric g^W on W , so that the condition (iii) of Definition 3.2 is satisfied. This is done in Section 13.

4.6. **The additivity of the index.** Suppose that Σ divides M into two open submanifolds M_1 and M_2 , so that $M_\Sigma = M_1 \sqcup M_2$. The metric g^{M_Σ} induces complete G -invariant Riemannian metrics g^{M_1}, g^{M_2} on M_1 and M_2 , respectively. Let $\mathcal{E}_i, \mathbf{v}_i$ ($i = 1, 2$) denote the restrictions of the Clifford module \mathcal{E}_Σ and the taming map \mathbf{v}_Σ to M_i . Then Theorem 4.4 implies the following

Corollary 4.7. $\chi_G^{an}(\mathcal{E}, \mathbf{v}) = \chi_G^{an}(\mathcal{E}_1, \mathbf{v}_1) + \chi_G^{an}(\mathcal{E}_2, \mathbf{v}_2)$.

Thus, we see that the index of non-compact manifolds is “*additive*”.

5. THE INDEX THEOREM

In this section we recall the definition of the *topological index* of a tamed Clifford module, cf. [1, 20, 14], and prove that it is equal to the analytical index.

5.1. **Transversally elliptic symbols.** Let M be a G -manifold and let $\pi : T^*M \rightarrow M$ be the projection. A G -equivariant map $\sigma \in \Gamma(T^*M, \text{Hom}(\pi^*\mathcal{E}^+, \pi^*\mathcal{E}^-))$ will be called a *symbol*.

Set

$$T_G^*M = \{ \xi \in T^*M : \langle \xi, v(\pi(\xi)) \rangle = 0 \text{ for all } \mathbf{v} \in \mathfrak{g} \}.$$

(Here, as usual, v denotes the vector field on M generated by the infinitesimal action of $\mathbf{v} \in \mathfrak{g}$). A symbol σ is called *transversally elliptic*⁴ if $\sigma(\xi) : \pi^*\mathcal{E}^+|_\xi \rightarrow \pi^*\mathcal{E}^-|_\xi$ is invertible for all $\xi \in T_G^*M$ outside of a compact subset of T_G^*M . A transversally elliptic symbol defines an element of the compactly supported G -equivariant K -theory $K_G(T_G^*M)$ of T_G^*M . Thus a construction of Atiyah [1] defines an index of such an element. We, next, recall the main steps of this construction.

5.2. The index of a transversally elliptic symbol on a compact manifold. Let $\sigma \in \Gamma(M, \text{Hom}(\pi^*\mathcal{E}^+, \pi^*\mathcal{E}^-))$ be a transversally elliptic symbol on a compact manifold M and let $P : \Gamma(M, \mathcal{E}^+) \rightarrow \Gamma(M, \mathcal{E}^-)$ be a G -invariant pseudo-differential operator, whose symbol coincides with σ .

For each irreducible representation $V \in \text{Irr } V$ let

$$\Gamma(M, \mathcal{E}^\pm)^V \approx \text{Hom}_G(V, \Gamma(M, \mathcal{E}^\pm)) \otimes V$$

be the isotopic component of $\Gamma(M, \mathcal{E}^\pm)$ corresponding to V . We denote by P^V the restriction of P to $\Gamma(M, \mathcal{E}^+)^V$ so that

$$P^V : \Gamma(M, \mathcal{E}^+)^V \rightarrow \Gamma(M, \mathcal{E}^-)^V.$$

It was shown by Atiyah [1] that, if M is compact, then the operator P^V is Fredholm, so that the index

$$\chi_G(P) := \sum_{V \in \text{Irr } G} (\dim \text{Ker } P^V - \dim \text{Coker } P^V) \cdot V \quad (5.1)$$

is defined. Moreover, the sum (5.1) depends only on the (homotopy class of the) symbol σ , but not on the choice of the operator P . Hence, we can define the index $\chi_G(\sigma)$ by $\chi_G(\sigma) := \chi_G(P)$.

5.3. The topological index of a transversally elliptic symbol on a non-compact manifold. Let now σ be a transversally elliptic symbol on a non-compact manifold M . In particular, this means that there exists an open relatively compact subset $U \subset M$ such that $\sigma(\xi)$ is invertible for all $\xi \in \pi^{-1}(M \setminus U)$.

Lemma 3.1 of [1] shows that there exists a transversally elliptic symbol $\tilde{\sigma} : \pi^*\tilde{\mathcal{E}}^+ \rightarrow \pi^*\tilde{\mathcal{E}}^-$ which represents the same element in $K_G(T_G^*M)$ as σ and such that the restrictions of the bundles $\tilde{\mathcal{E}}^\pm$ to $M \setminus U$ are trivial, and $\tilde{\sigma}|_{M \setminus U}$ is an identity.

Fix an open relatively compact subset $\tilde{U} \subset M$ which contains the closure of U . Let $j : \tilde{U} \hookrightarrow N$ be a G -equivariant embedding of \tilde{U} into a compact G -manifold N (such an embedding always exists, cf., for example, Lemma 3.1 of [14]).

The symbol $\tilde{\sigma}$ extends naturally to a transversally elliptic symbol $\tilde{\sigma}_N$ over N . The excision theorem 3.7 of [1] asserts that the index $\chi_G(\tilde{\sigma}_N)$ depends only on σ but not on the choices of $U, \tilde{U}, \tilde{\sigma}$ and j . One, thus, can define the *topological index* of σ by

$$\chi_G^{\text{top}}(\sigma) := \chi_G(\tilde{\sigma}_N).$$

⁴Note that our notion of transversally elliptic symbol is slightly more restrictive than the original Atiyah's definition, [1] (but is the same as in [20]). It coincides with the Atiyah's definition if M is compact.

5.4. The topological index of a tamed Clifford module. Suppose now $(\mathcal{E}, \mathbf{v})$ is a tamed Clifford module over a complete Riemannian manifold M . Clearly,

$$\sigma_{\mathcal{E}}(\xi) := \sqrt{-1} c(\xi) + \sqrt{-1} c(v) = \sqrt{-1} c(\xi + v)$$

defines a transversally elliptic symbol on M . We then define *topological index of $(\mathcal{E}, \mathbf{v})$* by

$$\chi_G^{\text{top}}(\mathcal{E}, \mathbf{v}) := \chi_G^{\text{top}}(\sigma_{\mathcal{E}}).$$

The main result of this paper is the following

Theorem 5.5. *For any tamed Clifford module $(\mathcal{E}, \mathbf{v})$ the analytic and topological indexes coincide*

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) = \chi_G^{\text{top}}(\mathcal{E}, \mathbf{v}).$$

The proof is given in Section 14. Here we only explain the main steps of the proof.

5.6. The sketch of the proof of Theorem 5.5. Let $U \subset M$ be a G -invariant open relatively compact set with smooth boundary which contains all the zeros of the vector field v . We endow U with a complete Riemannian metric and we denote by $(\mathcal{E}_U, \mathbf{v}_U)$ the induced tamed Clifford module over U . Combining Corollary 4.7 with Lemma 3.12, we obtain

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) = \chi_G^{\text{an}}(\mathcal{E}_U, \mathbf{v}_U).$$

Let \tilde{U} be an open relatively compact G -invariant subset of M , which contains the closure of U . Fix a G -equivariant embedding of \tilde{U} into a compact manifold N .

In Subsection 14.2, we extend \mathcal{E}_U to a graded vector bundle $\tilde{\mathcal{E}}_N = \tilde{\mathcal{E}}_N^+ \oplus \tilde{\mathcal{E}}_N^-$ over N and we extend the map $c(v)$ to a map $\bar{c} : \tilde{\mathcal{E}}^+ \rightarrow \tilde{\mathcal{E}}^-$, whose restriction to $N \setminus U$ is the identity map.

As in Subsection 4.2, define a Clifford module \mathcal{E}_U on U , which corresponds to a complete Riemannian metric of the form $g^U = \frac{1}{\alpha^2} g^M$. Fix a Clifford connection $\nabla^{\mathcal{E}_U}$ on \mathcal{E}_U and let f be an admissible function for $(\mathcal{E}_U, \mathbf{v}|_U, \nabla^{\mathcal{E}_U})$. We can and we will assume that the function

$$\bar{f}(x) = \begin{cases} 1/f(x), & x \in U; \\ 0, & x \notin U, \end{cases}$$

is continuous.

Let $A : \Gamma(N, \tilde{E}^+) \rightarrow \Gamma(N, \tilde{E}^+)$ denote an invertible positive-definite self-adjoint G -invariant second-order differential operator, whose symbol is equal to $|\xi|^2$. In Subsection 14.4, we show that the symbol of the transversally elliptic operator

$$P = \sqrt{-1} \bar{c} + \bar{f} \alpha^{-1} D_U^+ A^{-1/2}$$

is homotopic to $\sigma_{\mathcal{E}}$. Hence, $\chi_G^{\text{top}}(\mathcal{E}, \mathbf{v}) = \chi_G(P)$.

In Subsections 14.5 and 14.6 we use the deformation arguments to show that $\chi_G(P)$ is equal to the index of operator $\sqrt{-1} \bar{c} + \bar{f} \alpha^{-1} D_U^+$. Note that the later operator is not transversally elliptic. However, an explicit calculation made in Subsection 14.6 shows that its index is well defined and is equal to $\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v})$.

6. AN EXAMPLE: VECTOR BUNDLE

In this section we assume that G is a torus and present a formula for the index of a tamed Clifford module over a manifold M , which has a structure of the total space of a vector bundle $p : N \rightarrow F$. This formula was probably known for a very long time. Some particular cases can be found in [1, Lecture 6] and [20, Part II]. The general case was proven by Paradan [14, §5].

The results of this section will be used in the next section to obtain the extension of the equivariant index theorem of Atiyah-Segal-Singer to non-compact manifolds.

6.1. The setting. Let M be the total space of a vector bundle $p : N \rightarrow F$. Assume that the torus group G acts on M by linear transformations of the fibers and that it preserves only the zero section of the bundle.

Let g^M be a complete G -invariant Riemannian metric on M . Let $\mathbf{v} : M \rightarrow \mathfrak{g}$ be a taming map such that both vector fields on M induced by \mathbf{v} and by the composition $\mathbf{v} \circ p : N \rightarrow F$ do not vanish outside of F .

Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a G -equivariant \mathbb{Z}_2 -graded self-adjoint Clifford module over M .

6.2. The decomposition of $N^{\mathbb{C}}$. Let $N^{\mathbb{C}} \rightarrow F$ denote the complexification of the bundle $N \rightarrow F$. We identify F with the zero section of N . The element $\mathbf{v}(x) \in \mathfrak{g}$ ($x \in F$), acts on the fiber $N_x^{\mathbb{C}} := p^{-1}(x)$ of $N^{\mathbb{C}}$ by linear skew-adjoint transformations. Hence, the spectrum of the restriction of the operator $\sqrt{-1}\mathbf{v}(x)$ to each fiber $N^{\mathbb{C}}$ is real.

Since $G = T^n$ does not have fixed points outside of the zero section, the dimension of the fiber of $p : N \rightarrow F$ is even. Moreover, we can and we will choose a G -invariant complex structure $J : N \rightarrow N$ on the fibers of N , so that the restriction of $\sqrt{-1}\mathbf{v}(x)$ to the holomorphic space $N_x^{1,0} \subset N_x^{\mathbb{C}}$ has only positive eigenvalues.

6.3. The decomposition of \mathcal{E} . Let $T_{\text{vert}}M \subset TM$ denote the bundle of vectors tangent to the fibers of $p : N \rightarrow F$. Let $T_{\text{hor}}M$ be the orthogonal complement of $T_{\text{vert}}M$. Let $T_{\text{vert}}^*M, T_{\text{hor}}^*M$ be the dual bundles. We have an orthogonal direct sum decomposition $T^*M = T_{\text{hor}}^*M \oplus T_{\text{vert}}^*M$. Hence, the Clifford algebra of T^*M decomposes as a tensor product

$$C(T^*M) = C(T_{\text{hor}}^*M) \otimes C(T_{\text{vert}}^*M). \quad (6.1)$$

Consider the bundle $\Lambda^{\bullet}((N^{0,1})^*)$ of anti-holomorphic forms on N . The lift Λ of this bundle to M has a natural structure of a module over T_{vert}^*M and, in fact, is isomorphic to the space of “vertical spinors” on M , cf. [4, Ch. 3.2]. It follows from [4, Ch. 3], that the bundle \mathcal{E} decomposes into a (graded) tensor product

$$\mathcal{E} \simeq \mathcal{W} \otimes \Lambda$$

where \mathcal{W} is a G -equivariant \mathbb{Z}_2 -graded Hermitian vector bundle over M , on which $C(T_{\text{vert}}^*M)$ acts trivially. By [4, Prop. 3.27], there is a natural isomorphism

$$\text{End}_{C(T_{\text{vert}}^*M)}(\mathcal{W} \otimes \Lambda) \simeq \text{End}_{\mathbb{C}} \mathcal{W}, \quad (6.2)$$

The Clifford algebra $C(T_{\text{hor}}^*M)$ of T_{hor}^*M acts on \mathcal{E} and this action commutes with the action of $C(T_{\text{vert}}^*M)$. The isomorphisms (6.1), (6.2) define a G -equivariant action of $C(T_{\text{hor}}^*M)$ on \mathcal{W} .

Let $S((N^{1,0})^*) = \bigoplus_k S^k((N^{1,0})^*) \rightarrow F$ be the sum of the symmetric powers of the dual of the bundle $N^{1,0}$. It is endowed with a natural Hermitian metric (coming from the Riemannian metric on M) and with a natural action of G .

6.4. The bundle K_F . Let us define a bundle $K_F = \mathcal{W}|_F \otimes S((N^{1,0})^*)$. The group G acts on K_F and the subbundle of any given weight has finite dimension. In other words,

$$K_F = \bigoplus_{\alpha \in \mathcal{L}} \mathcal{E}_\alpha,$$

where α runs over the set of all integer weights $\mathcal{L} \simeq \mathbb{Z}^n$ of G and each \mathcal{E}_α is a finite dimensional vector bundle, on which G acts with weight α . Each \mathcal{E}_α is endowed with the action of the Clifford algebra of $T^*F \simeq T_{\text{hor}}^*M|_F$, induced by its action on $\mathcal{W}|_F$. It also possesses natural Hermitian metric and grading. Let D_α denote the Dirac operator associated to a Hermitian connection on \mathcal{E}_α . We will consider the (non-equivariant) index

$$\text{ind } D_\alpha = \dim \text{Ker } D_\alpha^+ - \dim \text{Ker } D_\alpha^-$$

of this operator. By the Atiyah-Singer index theorem

$$\text{ind } D_\alpha = \int_F \hat{A}(F) \cdot \text{ch}(\mathcal{E}_\alpha),$$

where $\text{ch}(\mathcal{E}_\alpha)$ is the Chern character of \mathcal{E}_α (cf. [4, §4.1]).

Theorem 6.5. *The index⁵ of the tamed Clifford module $(\mathcal{E}, \mathbf{v})$ is given by*

$$\chi_G(\mathcal{E}, \mathbf{v}) = \sum_{\alpha \in \mathcal{L}} \text{ind } D_\alpha \cdot V_\alpha = \sum_{\alpha \in \mathcal{L}} \left[\int_F \hat{A}(F) \cdot \text{ch}(\mathcal{E}_\alpha) \right] \cdot V_\alpha, \quad (6.3)$$

where V_α denotes the (one-dimensional) irreducible representation of G with weight α .

A K -theoretical proof of this theorem can be found in [14, §5]⁶. For the case when M is a Kähler manifold, this theorem was proven by Wu and Zhang [22] by a direct analytic calculation of $\text{Ker } D_{f_v}$. The method of Wu and Zhang works with minor changes for general manifolds. Note that our formula is simpler than the one in [22], because we had the freedom of choosing a convenient complex structure on N .

Remark 6.6. Since the action of $\sqrt{-1} \mathbf{v}(x)$ on $S((N^{1,0})^*)$ has only negative eigenvalues, there exists a constant $C > 0$, such that $\mathcal{E}_\alpha = 0$ if $\alpha(\mathbf{v}(x)) > C$ for all $x \in F \subset M$. It follows, that $\chi_G(\mathcal{E}, \mathbf{v})$ contains only representations with weights α , such that $\alpha(\mathbf{v}) \leq C$.

⁵We don't distinguish any more between the topological and analytic indexes in view of the index theorem 5.5.

⁶In [14], $\mathbf{v} = \mathbf{v} \circ p$. The general case follows from the fact that $(\mathcal{E}, \mathbf{v})$ is, obviously, cobordant to $(\mathcal{E}, \mathbf{v} \circ p)$

7. THE EQUIVARIANT INDEX THEOREM ON OPEN MANIFOLDS

In this section we present a generalization of the Atiyah-Segal-Singer equivariant index theorem to complete Riemannian manifolds. In particular, we obtain a new proof of the classical Atiyah-Segal-Singer equivariant index theorem for compact manifolds. Our proof is based on an analogue of Guillemin-Ginzburg-Karshon linearization theorem, which, roughly speaking, states that a tamed G -manifold (where G is a torus) is cobordant to the normal bundle to the fixed point set for the G action. The approach of this section is an analytic counterpart of the K -theoretic study in [1, Lect. 6], [20, Part II] and [14, §4].

Throughout the section *we assume that G is a torus.*

7.1. The linearization theorem. Suppose (M, \mathbf{v}) is a tamed G -manifold and let $F \subset M$ be the set of points fixed by the G -action. Then the vector field v on M vanishes on F . It follows that F is compact. Hence, it is a disjoint union of compact smooth manifolds F_1, \dots, F_k . Let N_i denote the normal bundle to F_i in M and let N be the disjoint union of N_i . Let $p : N \rightarrow F$ be the natural G -invariant projection. In this section we do not distinguish between the vector bundle N and its total space.

Let $\mathbf{v} : M \rightarrow \mathfrak{g}$ be a taming map. Let $\mathbf{v}_N : N \rightarrow \mathfrak{g}$ be a G -equivariant map, such that $\mathbf{v}_N|_F \equiv \mathbf{v}|_F$ (in applications, we will set $\mathbf{v}_N = \mathbf{v} \circ p : N \rightarrow \mathfrak{g}$). *We assume that the vector field v on M induced by \mathbf{v} , the vector field v_N on N induced by \mathbf{v}_N and the vector field induced on N by $\mathbf{v} \circ p : N \rightarrow \mathfrak{g}$ do not vanish outside of F .* (The last condition is equivalent to the statement that v has a zero of first order on F).

The bundles $TN|_F$ and $TM|_F$ over F are naturally isomorphic. Hence, the Riemannian metric on M induces a metric on $TN|_F$. Fix a complete G -invariant Riemannian metric on N , whose restriction to $TN|_F$ coincides with this metric. Then (N, \mathbf{v}_N) is a tamed G -manifold.

The following theorem is an analogue of Karshon's linearization theorem [12], [9, Ch. 4].

Linearization Theorem 7.2. *Suppose (M, \mathbf{v}) is a tamed G -manifold, such that the vector field induced by \mathbf{v} on M and the vector fields induced by \mathbf{v}_N and $\mathbf{v} \circ p$ on N do not vanish outside of F . Suppose $\mathcal{E}, \mathcal{E}_N$ are G -equivariant self-adjoint \mathbb{Z}_2 -graded Clifford modules over M and N , respectively. Assume that $\mathcal{E}_N|_F \simeq \mathcal{E}|_F$ as Hermitian modules over the Clifford algebra of $T^*M|_F$. Then the tamed Clifford module $(\mathcal{E}, \mathbf{v})$ is cobordant to $(\mathcal{E}_N, \mathbf{v}_N)$.*

The proof is very similar to the proof of the gluing formula, cf. Section 13. We present only the main idea of the proof. The interested reader can easily fill the details.

7.3. The idea of the proof of the Linearization theorem. Let V be a tubular neighborhood of F in M , which is G -equivariantly diffeomorphic to N .

Consider the product $M \times [0, 1]$, and the set

$$Z := \{ (x, t) \in M \times [0, 1] : t \leq 1/3, x \notin V \}.$$

Set $W := (M \times [0, 1]) \setminus Z$. Then W is an open G -manifold, whose boundary is diffeomorphic to the disjoint union of $N \simeq V \times \{0\}$ and $M \simeq M \times \{1\}$. Essentially, W is the required cobordism. However, we have to be accurate in defining a complete Riemannian metric g^W on W , so that

the condition (iii) of Definition 3.2 is satisfied. This can be done in more or less the same way as in Section 13.

7.4. The equivariant index theorem. We now apply the construction of Section 6 to the normal bundle $N_i \rightarrow F_i$. In particular, we choose a G -invariant complex structure on N and consider the infinite dimensional G -equivariant vector bundle $K_{F_i} = \mathcal{E}|_{F_i} \otimes S((N_i^{1,0})^*)$. We write $K_{F_i} = \bigoplus_{\alpha \in \mathcal{L}} \mathcal{E}_{i,\alpha}$, where α runs over the set of all integer weights $\mathcal{L} \simeq \mathbb{Z}^n$ of G and each $\mathcal{E}_{i,\alpha}$ is a finite dimensional vector bundle on which G acts with weight α . Then, cf. Section 6, each $\mathcal{E}_{i,\alpha}$ has a natural structure of a Clifford module over F_i . Let $D_{i,\alpha}$ denote the corresponding Dirac operator. The main result of this section is the following analogue of the Atiyah-Segal-Singer equivariant index theorem

Theorem 7.5. *Suppose the map (M, \mathbf{v}) is a tamed G -manifold, such that both vector fields on M induced by \mathbf{v} and by $\mathbf{v} \circ p$ do not vanish outside of F . Suppose \mathcal{E} is a \mathbb{Z}_2 -graded self-adjoint Clifford module over M . Then, using the notation introduced above, we have*

$$\chi_G(\mathcal{E}, \mathbf{v}) = \sum_{\alpha \in \mathcal{L}} \left(\sum_{i=1}^k \text{ind } D_{i,\alpha} \right) \cdot V_\alpha = \sum_{\alpha \in \mathcal{L}} \left(\sum_{i=1}^k \int_F \hat{A}(F_i) \cdot \text{ch}(\mathcal{E}_{i,\alpha}) \right) \cdot V_\alpha, \quad (7.1)$$

where V_α denotes the (one-dimensional) irreducible representation of G with weight α .

The theorem is an immediate consequence of the cobordism invariance of the index (Theorem 3.7) and the linearization theorem 7.2.

7.6. The classical Atiyah-Segal-Singer theorem. Suppose now that M is a *compact* manifold. Then the index $\chi_G(D, \mathbf{v})$ is independent of \mathbf{v} and is equal to the index representation $\chi_G(D) = \text{Ker } D^+ \ominus \text{Ker } D^-$. Theorem 7.5 reduces in this case to the classical Atiyah-Segal-Singer equivariant index theorem [2]. We, thus, obtain a new geometric proof of this theorem.

8. AN ADMISSIBLE FUNCTION ON A MANIFOLD WITH BOUNDARY

In the proof of Theorem 3.7 we will need a notion of admissible function on a cobordism, which extends Definition 2.6. In this section we define this notion and prove the existence of such a function. In particular, we will prove Lemma 2.7 about the existence of an admissible function on a manifold without boundary.

8.1. Let $(\mathcal{E}, \mathbf{v})$ be a tamed Clifford module over a complete G -manifold M . Let (W, \mathbf{v}_W, ϕ) be a cobordism between (M, \mathbf{v}) and the empty set, cf. Definition 3.2. In particular, W is a complete G -manifold with boundary and ϕ is a G -equivariant metric preserving diffeomorphism between a neighborhood U of $\partial W \simeq M$ and the product $M \times [0, \varepsilon)$.

Let $\pi : M \times [0, \varepsilon) \rightarrow M$ be the projection. A G -invariant Clifford connection $\nabla^\mathcal{E}$ on \mathcal{E} induces a connection $\nabla^{\pi^*\mathcal{E}}$ on the pull-back $\pi^*\mathcal{E}$, such that

$$\nabla_{(u,a)}^{\pi^*\mathcal{E}} := \pi^*\nabla_u^\mathcal{E} + a \frac{\partial}{\partial t}, \quad (u, a) \in TM \times \mathbb{R} \simeq T(M \times [0, \varepsilon)). \quad (8.1)$$

Let $(\mathcal{E}_W, \mathbf{v}_W, \psi)$ be a cobordism between $(\mathcal{E}, \mathbf{v})$ and the unique Clifford module over the empty set, cf. Definition 3.5. In particular, $\psi : \mathcal{E}_W|_U \rightarrow \pi^* \mathcal{E}$ is a G -equivariant isometry. Let $\nabla^{\mathcal{E}_W}$ be a G -invariant connection on \mathcal{E}_W , such that $\nabla^{\mathcal{E}}|_{\phi^{-1}(M \times [0, \varepsilon/2])} = \psi^{-1} \circ \nabla^{\pi^* \mathcal{E}} \circ \psi$.

Definition 8.2. A smooth G -invariant function $f : W \rightarrow [0, \infty)$ is an admissible function for the cobordism $(\mathcal{E}_W, \mathbf{v}_W, \nabla^{\mathcal{E}_W})$, if it satisfies (2.5) and there exists a function $h : M \rightarrow [0, \infty)$ such that $f(\phi^{-1}(y, t)) = h(y)$ for all $y \in M, t \in [0, \varepsilon/2)$.

Lemma 8.3. Suppose h is an admissible function for $(\mathcal{E}_M, \mathbf{v}, \nabla^{\mathcal{E}})$. Then there exists an admissible function f on $(\mathcal{E}_W, \mathbf{v}_W, \nabla^{\mathcal{E}_W})$ such that the restriction $f|_M = h$.

Proof. Consider a smooth function $r : W \rightarrow [0, \infty)$ such that

- $|dr(x)| \leq 1$, for all $x \in W$, and $\lim_{x \rightarrow \infty} r(x) = \infty$;
- there exists a smooth function $\rho : M \rightarrow [0, \infty)$, such that $r(\phi^{-1}(y, t)) = \rho(y)$ for all $y \in M, t \in [0, 3\varepsilon/4)$.

Then the set $\{x \in W : r(x) = t\}$ is compact for all $t \geq 0$. Let v denote the vector field induced by \mathbf{v}_W on W . Recall that the function ν is defined in (2.4). Let $a : [0, \infty) \rightarrow [0, \infty)$ be a smooth strictly increasing function, such that

$$a(t) \geq 2 \max \left\{ \frac{\nu(x)}{|v(x)|^2} : r(x) = t \right\} + t + 1; \quad t \gg 0.$$

Let $b : [0, \infty) \rightarrow [0, \infty)$ be a smooth function, such that

$$0 < b(t) \leq \min \left\{ \frac{a'(t)}{a(t)^2}; \frac{1}{t^2} \right\}.$$

Set

$$g(t) = \left(\int_t^\infty b(s) ds \right)^{-2}.$$

The integral converges, since $b(s) \leq 1/s^2$. Moreover,

$$g(t)^{1/2} \geq a(t) > t; \quad g'(t) = 2g^{3/2}b > 0, \quad t \gg 0. \quad (8.2)$$

Let $\alpha : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that

- $\alpha(t) = 0$ for $|t| \geq 2\varepsilon/3$;
- $\alpha(t) = 1$ for $|t| \leq \varepsilon/3$.

Let $C = \max\{|\alpha'(t)| : t \in \mathbb{R}\}$ and let $\beta : W \rightarrow [0, 1]$ be a smooth function, such that $\beta(\phi^{-1}(y, t)) = \alpha(t)$ for $y \in M, t \in [0, \varepsilon)$ and $\beta(x) = 0$ for $x \notin U$. Then $|d\beta| \leq C$.

Recall that h is an admissible function for $(\mathcal{E}, \mathbf{v}, \nabla^{\mathcal{E}})$. This function induces a function on $U \simeq M \times [0, \varepsilon)$, which, by a slight abuse of notation, we will also denote by h .

Set

$$f(x) := \begin{cases} \beta(x)h(x) + (1 - \beta(x))g(r(x)), & x \in M \times [0, \varepsilon), \\ g(r(x)), & x \notin M \times [0, \varepsilon). \end{cases}$$

Clearly, $f(\phi^{-1}(y, t)) = h(y)$ for all $y \in M, t \in [0, \varepsilon/3)$.

We have to show that $\frac{f^2|v|^2}{|df||v|+f\nu+1}$ tends to infinity as $x \rightarrow \infty$. Consider, first, the case $x \notin U$. Then $f(x) = g(r(x))$ and $|df| = g'|dr| \leq g'$. Hence, from the definition of the functions a we get

$$\begin{aligned} \frac{|df||v| + f\nu + 1}{|v|^2} &\leq \frac{g'|v| + g\nu + 1}{|v|^2} \\ &\leq (g'(r) + g(r) + 1) \frac{\nu(x)}{|v(x)|^2} \leq a(r)(g'(r) + g(r) + 1). \end{aligned} \quad (8.3)$$

From (8.3) and (8.2), we obtain

$$\begin{aligned} \frac{f^2|v|^2}{|df||v| + f\nu + 1} &\geq \frac{g^2}{a(g' + g + 1)} \geq \frac{g^{3/2}}{2g^{3/2}b + g + 1} \\ &= \frac{1}{2b + g^{-1/2} + g^{-3/2}} \geq \frac{1}{2r^{-2} + r^{-1} + r^{-3}} \rightarrow \infty \end{aligned} \quad (8.4)$$

as $x \rightarrow \infty$. Note that (8.4) holds even if $x \in U$, though in this case $g(r(x)) \neq f(x)$.

Consider now the case $x \in U$. Then

$$\begin{aligned} |df||v| + f\nu + 1 &\leq \beta(|dh||v| + h\nu) + (1 - \beta)(g'(r)|dr||v| + g\nu) + |d\beta||h - g||v| + 1 \\ &\leq \beta(|dh||v| + h\nu) + (g'|v| + g\nu) + C(h + g)|v| + 1 \\ &\leq 2(1 + C) \max \left\{ |dh||v| + h\nu + 1; g'|v| + g\nu + 1 \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{f^2|v|^2}{|df||v| + f\nu + 1} &\geq \frac{f^2|v|^2}{2(1 + C) \max \left\{ |dh||v| + h\nu + 1; g'|v| + g\nu + 1 \right\}} \\ &\geq \frac{1}{2(1 + C)} \max \left\{ \frac{\beta^2 h^2 |v|^2}{|dh||v| + h\nu + 1}; \frac{(1 - \beta)^2 g^2 |v|^2}{g'|v| + g\nu + 1} \right\}. \end{aligned}$$

When $x \rightarrow \infty, x \in U$, the expression $\frac{h^2|v|^2}{|dh||v|+h\nu+1}$ tends to infinity by definition of h , while $\frac{g^2|v|^2}{g'|v|+g\nu+1}$ tends to infinity by (8.4).

Lemma 8.3 is proven. \square

8.4. Proof of Lemma 2.7. Lemma 2.7 follows from Lemma 8.3 by setting $W = M$ (so that $\partial W = \emptyset$). \square

9. PROOF OF THEOREM 2.9.1

9.1. Calculation of D_{fv} . Let f be an admissible function and set $u = fv$. Consider the operator

$$A_u = \sum c(e_i) c(\nabla_{e_i}^{\text{LC}} u) : \mathcal{E} \rightarrow \mathcal{E}, \quad (9.1)$$

where $\bar{e} = \{e_1 \dots e_n\}$ is an orthonormal frame of $TM \simeq T^*M$ and ∇^{LC} is the Levi-Civita connection on TM . One easily checks that A_u is independent of the choice of \bar{e} (it follows, also, from Lemma 9.2 below).

The proof of Theorem 2.9.1 is based on the following

Lemma 9.2. *Let D_u be the deformed Dirac operator defined in (2.6), then*

$$D_u^2 = D^2 + |u|^2 + \sqrt{-1}A_u - 2\sqrt{-1}\nabla_u^\mathcal{E}. \quad (9.2)$$

The proof of the lemma is a straightforward calculation.

9.3. Proof of Theorem 2.9. Since the operator D_u is self-adjoint, $\text{Ker } D_u = \text{Ker } D_u^2$. Hence, it is enough to show that each irreducible representation of G appears in $\text{Ker } D_u^2$ with finite multiplicity.

Fix $V \in \text{Irr } G$ and let

$$\Gamma(M, \mathcal{E})^V \simeq \text{Hom}_G(V, \Gamma(M, \mathcal{E})) \otimes V \quad (9.3)$$

be the isotypic component of $\Gamma(M, \mathcal{E})$ corresponding to V . The irreducible representation V appears in $\text{Ker } D_u^2$ with the multiplicity equal to the dimension of the kernel of the restriction of D_u^2 to the space $\Gamma(M, \mathcal{E})^V$. We will now use (9.2) to estimate this restriction from below.

Note, first, that, since $\|c(v)\| = |v|$ and $\|c(e_i)\| = 1$, we have

$$\|A_u\| \leq \sum_i \|\nabla_{e_i}^{\text{LC}} u\| \leq C \left(|df| |v| + f \|\nabla^{\text{LC}} v\| \right), \quad (9.4)$$

for some constant $C > 0$.

Using the definition (2.3) of $\mu^\mathcal{E}$, we obtain $\nabla_u^\mathcal{E} = \mathcal{L}_u^\mathcal{E} + \mu^\mathcal{E}(\mathbf{u})$. For every $\mathbf{a} \in \mathfrak{g}$, the operator $\mathcal{L}_\mathbf{a}^\mathcal{E}$ is bounded on $\Gamma(M, \mathcal{E})^V$. Hence, there exists a constant c_V such that

$$\|\mathcal{L}_\mathbf{u}^\mathcal{E}|_{\Gamma(M, \mathcal{E})^V}\| \leq c_V |\mathbf{u}|.$$

Thus, on $\Gamma(M, \mathcal{E})^V$ we have

$$\|\nabla_u^\mathcal{E}\| \leq \|\mathcal{L}_\mathbf{u}^\mathcal{E}\| + \|\mu^\mathcal{E}(\mathbf{u})\| = f \left(\|\mathcal{L}_\mathbf{v}^\mathcal{E}\| + \|\mu^\mathcal{E}(\mathbf{v})\| \right) \leq f \left(c_V |\mathbf{v}| + \|\mu^\mathcal{E}(\mathbf{v})\| \right). \quad (9.5)$$

Combining, (9.2), (9.4) and (9.5), we obtain

$$D_u^2|_{\Gamma(M, \mathcal{E})^V} \geq D^2|_{\Gamma(M, \mathcal{E})^V} + f^2 |v|^2 - \lambda_V \left(|df| |v| + f \left(|\mathbf{v}| + \|\mu^\mathcal{E}(\mathbf{v})\| + \|\nabla^{\text{LC}} v\| \right) \right),$$

where $\lambda_V = \max\{1, c_V, C\}$. It follows now from (2.5), that there exists a real valued function $r_V(x)$ on M such that $\lim_{x \rightarrow \infty} r_V(x) = +\infty$ and on $\Gamma(M, \mathcal{E})^V$ we have

$$D_u^2|_{\Gamma(M, \mathcal{E})^V} \geq D^2|_{\Gamma(M, \mathcal{E})^V} + r_V(x). \quad (9.6)$$

It is well known (cf., for example, [19, Lemma 6.3]) that the spectrum of $D^2 + r_V(x)$ is discrete. Hence, since the operator $D^2|_{\Gamma(M, \mathcal{E})^V} + r_V(x)$ is bounded below, the inequality (9.6) implies that the spectrum of the restriction of D_u^2 to $\Gamma(M, \mathcal{E})^V$ is also discrete (cf. Prop. 3.1 of [3, Supplement 1] or Problem 1 of [16, Ch. XIII]). \square

10. PROOF OF THEOREM 3.7

By Remark 3.6, it is enough to prove Theorem 3.7 in the case, when W is a cobordism between a tamed G -manifold (M, \mathbf{v}) and an empty set, which we shall henceforth assume.

10.1. The Clifford module structure on $\tilde{\mathcal{E}}$. Let us consider two anti-commuting actions (left and right action) of the Clifford algebra of \mathbb{R} on the exterior algebra $\Lambda^\bullet \mathbb{C} = \Lambda^0 \mathbb{C} \oplus \Lambda^1 \mathbb{C}$, given by the formulas

$$c_L(t)\omega = t \wedge \omega - \iota_t \omega; \quad c_R(t)\omega = t \wedge \omega + \iota_t \omega. \quad (10.1)$$

Note, that $c_L(t)^2 = -t^2$, while $c_R(t)^2 = t^2$. In the terminology of [4], these two actions correspond to the bilinear forms $(t, s) = ts$ and $(t, s) = -ts$ respectively.

We will use the notation of Subsection 3.8. In particular, \tilde{W} is the manifold obtained from W by attaching cylinders. We denote by \mathcal{E}'_W the extension of the bundle \mathcal{E}_W to \tilde{W} and we set $\tilde{\mathcal{E}} = \mathcal{E}'_W \otimes \Lambda^\bullet \mathbb{C}$.

Define a map $\tilde{c} : T^* \tilde{W} \rightarrow \text{End } \tilde{\mathcal{E}}$ by the formula

$$\tilde{c}(v) := \sqrt{-1} c(v) \otimes c_L(1), \quad v \in T^* \tilde{W}, \quad (10.2)$$

and set

$$\tilde{\mathcal{E}}^+ := \mathcal{E}'_W \otimes \Lambda^0 \mathbb{C}; \quad \tilde{\mathcal{E}}^- := \mathcal{E}'_W \otimes \Lambda^1 \mathbb{C}. \quad (10.3)$$

By a direct computation, one easily checks that (10.2), (10.3) define a structure of a self-adjoint \mathbb{Z}_2 -graded Clifford module on $\tilde{\mathcal{E}}$.

10.2. The Dirac operator on \tilde{W} . Recall that $\phi : U \rightarrow M \times [0, \varepsilon)$ is a diffeomorphism, defined in Definition 3.2, and that ψ is an isomorphism between the restriction of \mathcal{E}_W to U and the vector bundle $\pi^* \mathcal{E}$ induced on $M \times [0, \varepsilon)$ by \mathcal{E} , cf. Definition 3.5. The connection $\nabla^\mathcal{E}$ on \mathcal{E} induces a connection $\nabla^{\pi^* \mathcal{E}}$ on $\pi^* \mathcal{E}$, cf. (8.1). Choose a G -invariant Clifford connection on \mathcal{E}_W , whose restriction to $\phi^{-1}(M \times [0, \varepsilon/2))$ coincides with $\nabla^{\pi^* \mathcal{E}}$. This connection extends naturally to a G -invariant Clifford connection $\nabla^{\tilde{\mathcal{E}}}$ on $\tilde{\mathcal{E}}$.

Let \tilde{D} denote the Dirac operator on $\tilde{\mathcal{E}}$ corresponding to the Clifford connection $\nabla^{\tilde{\mathcal{E}}}$. We will need an explicit formula for the restriction of this operator to the cylinder $M \times (0, \infty)$. Let us introduce some notation. Let $t : M \times (0, \infty) \rightarrow (0, \infty)$ be the projection. We can and we will view t as a real valued function on the cylinder $M \times (0, \infty)$, so that $dt \in T^*(M \times (0, \infty))$. Note that $e_0 := \text{grad } t \in T(M \times (0, \infty))$ is the unit vector tangent to the fibers of the projection $\pi : M \times (0, \infty) \rightarrow M$. To simplify the notation, we denote

$$\gamma := c(dt) \otimes 1, \quad \frac{\partial}{\partial t} = \nabla_{e_0}^{\tilde{\mathcal{E}}}.$$

Let $t^* D$ denote the pull-back of the operator $D : \Gamma(M, \mathcal{E}) \rightarrow \Gamma(M, \mathcal{E})$ to $M \times (0, \infty)$. Then

$$\tilde{D}|_{M \times (0, \infty)} = \sqrt{-1} \left(t^* D + \gamma \frac{\partial}{\partial t} \right) \otimes c_L(1). \quad (10.4)$$

10.3. The operator D_a . Let f be an admissible function on M . Fix an admissible function on W whose restriction to M equals f , cf. Lemma 8.3. By a slight abuse of notation, we will denote this function by the same letter f . Also, to simplify the notation, we will denote the natural extension of f and \mathbf{v} to \tilde{W} by the same letters f, \mathbf{v} . Set

$$\tilde{D}_{fv} = \tilde{D} + \sqrt{-1} \tilde{c}(fv).$$

Let $s : \mathbb{R} \rightarrow [0, \infty)$ be a smooth function such that $s(t) = t$ for $|t| \geq 1$, and $s(t) = 0$ for $|t| \leq 1/2$. Consider the map $p : \tilde{W} \rightarrow \mathbb{R}$ such that

$$\begin{aligned} p(y, t) &= s(t), & \text{for } (y, t) \in M \times (0, \infty); \\ p(x) &= 0, & \text{for } x \in W. \end{aligned}$$

Clearly, p is a smooth function and the differential dp is uniformly bounded on \tilde{W} .

By a slight abuse of notation, we will write $c_L(s)$ and $c_R(s)$ for the operators $1 \otimes c_L(s)$ and $1 \otimes c_R(s)$, respectively. Note, that the operator $c_R(a)$ anti-commutes with \tilde{D}_{fv} , for all $a \in \mathbb{R}$. Set

$$\mathbf{D}_a = \tilde{D}_{fv} - c_R(p(x) - a), \quad a \in \mathbb{R}. \quad (10.5)$$

When restricted to the cylinder, $M \times (0, \infty)$ the bundle $\tilde{\mathcal{E}}$ is equal to $p^*\mathcal{E} \otimes \Lambda^\bullet \mathbb{C}$. Let

$$\Pi_0 : \tilde{\mathcal{E}} \rightarrow p^*\mathcal{E} \otimes \Lambda^0 \mathbb{C}; \quad \Pi_1 : \tilde{\mathcal{E}} \rightarrow p^*\mathcal{E} \otimes \Lambda^1 \mathbb{C}$$

be the projections.

Lemma 10.4. $\mathbf{D}_a^2 = \tilde{D}_{fv}^2 - B + |p(x) - a|^2$, where $B : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ is a uniformly bounded bundle map, whose restriction to $M \times (1, \infty)$ is equal to $\sqrt{-1} \gamma(\Pi_1 - \Pi_0)$, and whose restriction to W vanishes.

Proof. Note, first, that $p(x) - a \equiv -a$ on W . Thus, since $c_R(a)$ anti-commutes with \tilde{D}_{fv} , we have $\mathbf{D}_a^2|_W = \tilde{D}_{fv}^2|_W + a^2$. Hence, the identity of the lemma holds, when restricted to W .

We now consider the restriction of \mathbf{D}_a^2 to the cylinder $M \times (0, \infty)$. Recall that $t : M \times (0, \infty) \rightarrow (0, \infty)$ denotes the projection and that the function $s : \mathbb{R} \rightarrow [0, \infty)$ was defined in Subsection 10.3. Then

$$c_R(p(x) - a) = (s(t(x)) - a) c_R(1).$$

Using (10.4), we obtain

$$\tilde{D}_{fv}|_{M \times (0, \infty)} = \sqrt{-1} \left(t^* D + \sqrt{-1} c(fv) \right) c_L(1) + \sqrt{-1} \gamma c_L(1) \frac{\partial}{\partial t}.$$

The operators γ and $t^* D + \sqrt{-1} c(fv)$ commute with $(s(t) - a) c_R(1)$. Also the operators c_L and c_R anti-commute. Hence, we obtain

$$\begin{aligned} \mathbf{D}_a^2 &= \tilde{D}_{fv}^2 - \sqrt{-1} \gamma c_L(1) \frac{\partial}{\partial t} (s(t) - a) c_R(1) \\ &\quad - \sqrt{-1} \gamma (s(t) - a) c_R(1) c_L(1) \frac{\partial}{\partial t} + \left((s(t) - a) c_R(1) \right)^2 \\ &= \tilde{D}_{fv}^2 + \sqrt{-1} s' \gamma c_L(1) c_R(1) + |t - a|^2. \end{aligned}$$

Since $c_L(1) c_R(1) = \Pi_1 - \Pi_0$, it follows, that the statement of Lemma 10.4 holds with $B = s' \sqrt{-1} \gamma (\Pi_1 - \Pi_0)$. Since $s' = 1$ on $M \times (1, \infty)$, the restriction of B to this cylinder equals $\sqrt{-1} \gamma (\Pi_1 - \Pi_0)$. Finally, since s' is uniformly bounded on \tilde{W} , so is the bundle map $B : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$. \square

Since \mathbf{D}_a^2 is a G -invariant operator, G acts on $\text{Ker } \mathbf{D}_a^2$.

Proposition 10.5. *Each irreducible representation V of G appears in $\text{Ker } \mathbf{D}_a^2$ with finite multiplicity.*

Proof. We shall use the notation introduced in Subsection 9.3. In particular, $\Gamma(\tilde{W}, \tilde{\mathcal{E}})^V$ denotes the isotopic component of $\Gamma(\tilde{W}, \tilde{\mathcal{E}})$, corresponding to an irreducible representation V of G . As in Subsection 9.3, it is enough to prove that the spectrum of the restriction of \mathbf{D}_a^2 to $\Gamma(\tilde{W}, \tilde{\mathcal{E}})^V$ is discrete.

The arguments of Subsection 9.3 show that there exist a constant $c > 0$ and a smooth function $r_V : \tilde{W} \rightarrow [-c, \infty)$ such that

$$\tilde{D}_{fv}^2 \geq \tilde{D}^2 + r_V(x) \quad (10.6)$$

on $\Gamma(\tilde{W}, \tilde{\mathcal{E}})^V$ and the following 2 conditions hold

- $r_V(x) \rightarrow +\infty$ as $x \rightarrow \infty$ and $x \in W$;
- $r_V(y, t) \rightarrow +\infty$ uniformly in $t \in [0, \varepsilon)$, as $y \in M$ and $y \rightarrow \infty$.

Let $\|B(x)\|$, $x \in \tilde{W}$ denote the norm of the bundle map $B_x : \tilde{\mathcal{E}}_x \rightarrow \tilde{\mathcal{E}}_x$ and let $\|B\|_\infty = \sup_{x \in \tilde{W}} \|B(x)\|$. Set

$$R_V(x) := r_V(x) + |p(x) - a|^2 - \|B\|_\infty. \quad (10.7)$$

Then $R_V(x) \rightarrow +\infty$ as $\tilde{W} \ni x \rightarrow \infty$. Also, by Lemma 10.4 and (10.6), we have

$$\mathbf{D}_a^2|_{\Gamma(\tilde{W}, \tilde{\mathcal{E}})^V} \geq \tilde{D}^2 + R_V(x). \quad (10.8)$$

By [19, Lemma 6.3]), the spectrum of $\tilde{D}^2 + R_V(x)$ is discrete. Hence, (10.8) implies that so is the spectrum of the restriction of \mathbf{D}_a^2 to $\Gamma(\tilde{W}, \tilde{\mathcal{E}})^V$. \square

10.6. The index of \mathbf{D}_a . If V is an irreducible representation of G , we denote by $\mathbf{D}_a^{V, \pm}$ the restriction of \mathbf{D}_a to the space $\Gamma(\tilde{W}, \tilde{\mathcal{E}}^\pm)^V$. It follows from Proposition 10.5, that $\mathbf{D}_a^{V, \pm}$ is a Fredholm operator. In particular, all the irreducible representations of G appear in $\text{Ker } \mathbf{D}_a$ with finite multiplicities. Hence, we can define the index $\chi_G(\mathbf{D}_a)$ using (2.1), or, equivalently, by the formula

$$\chi_G(\mathbf{D}_a) := \sum_{V \in \text{Irr } G} \left(\dim \text{Ker } \mathbf{D}_a^{V, +} - \dim \text{Ker } \mathbf{D}_a^{V, -} \right) V. \quad (10.9)$$

Proposition 10.7. $\chi_G(\mathbf{D}_a) = 0$ for all $a \in \mathbb{R}$.

Proof. Each summand in (10.9) is the index of the operator $\mathbf{D}_a^{V, +}$. Thus, since

$$\mathbf{D}_a^{V, +} - \mathbf{D}_b^{V, +} = c_R(b - a) : L_2(\tilde{W}, \tilde{\mathcal{E}}) \rightarrow L_2(\tilde{W}, \tilde{\mathcal{E}})$$

is bounded operator depending continuously on $a, b \in \mathbb{R}$, the index $\chi_G(\mathbf{D}_a)$ is independent of a .

Therefore, it is enough to prove the proposition for one particular value of a . Recall that the norm $\|B\|_\infty$ was defined in the proof of Proposition 10.5. Choose $a \ll 0$ such that $a^2 > \|B\|_\infty$. It follows now from (10.7) and (10.8), that $\mathbf{D}_a^2 > 0$, so that $\text{Ker } \mathbf{D}_a^2 = 0$. Hence, $\chi_G(\mathbf{D}_a^2) = 0$. \square

Theorem 3.7 follows now from Proposition 10.7 and the following

Theorem 10.8. $\chi_G(\mathbf{D}_a) = \chi_G(D_{fv})$ for $a \gg 0$.

The proof of the theorem occupies the next section.

11. PROOF OF THEOREM 10.8

11.1. The plan of the proof. We consider an operator \mathbf{D}^{mod} on the cylinder $M \times \mathbb{R}$, with the following property: Let $\mathbf{D}_a^{\text{mod}}$, $a \in \mathbb{R}$ denote the operator obtained from \mathbf{D}^{mod} by the shift $T_a : (x, t) \rightarrow (x, t + a)$ (see Subsection 11.4 for a precise definition). Then the restrictions of $\mathbf{D}_a^{\text{mod}}$ and \mathbf{D}_a to the cylinder $M \times (1, \infty)$ coincide. Following Shubin [18], we call \mathbf{D}^{mod} the *model operator*.

In Lemma 11.3, we show that $\chi_G(D_{fv}) = \chi_G(\mathbf{D}_a^{\text{mod}})$ for all $a \in \mathbb{R}$.

The explicit formula for \mathbf{D}_a^2 , obtained in Lemma 10.4, shows that the restriction of this operator to the compliment of $M \times (1, \infty)$ becomes “very large” as $a \rightarrow \infty$. It follows that the eigenfunctions of \mathbf{D}_a are concentrated on $M \times (1, \infty)$ for large a . Hence, the kernel of \mathbf{D}_a can be estimated using the calculations on this cylinder, i.e., in terms of $\mathbf{D}_a^{\text{mod}}$. This is done in Proposition 11.6. Using this proposition it is easy to show that $\chi_G(\mathbf{D}_a) = \chi_G(\mathbf{D}_a^{\text{mod}})$ for large a , cf. Subsection 11.7.

11.2. The model operator on the cylinder. The restriction of $\tilde{\mathcal{E}}$ to $M \times (0, \infty)$ extends naturally to a Hermitian vector bundle over $M \times \mathbb{R}$, which we will also denote by $\tilde{\mathcal{E}}$. If $t : M \times \mathbb{R} \rightarrow \mathbb{R}$ denotes the projection, then $\tilde{\mathcal{E}} \simeq t^* \mathcal{E} \otimes \Lambda^\bullet \mathbb{C}$. Define the Clifford module structure and the grading on $\tilde{\mathcal{E}}$ using (10.2), (10.3).

Let $\tilde{D}, t^* D : \Gamma(M \times \mathbb{R}, \tilde{\mathcal{E}}) \rightarrow \Gamma(M \times \mathbb{R}, \tilde{\mathcal{E}})$ be, correspondingly, the Dirac operator on $\tilde{\mathcal{E}}$ and the pull-back of D . Using the notation introduced in Subsection 10.2, we can write

$$\tilde{D} = \sqrt{-1} \left(t^* D + \gamma \frac{\partial}{\partial t} \right) \otimes c_L(1).$$

Set

$$\mathbf{D}^{\text{mod}} := \tilde{D} + \sqrt{-1} \tilde{c}(fv) - c_R(t(x)) : \Gamma(M \times \mathbb{R}, \tilde{\mathcal{E}}) \rightarrow \Gamma(M \times \mathbb{R}, \tilde{\mathcal{E}}). \quad (11.1)$$

We will refer to \mathbf{D}^{mod} as the *model operator*, cf. [18]. This is a G -invariant elliptic operator. Moreover, it follows from Proposition 10.5 that the index of \mathbf{D}^{mod} is well defined. To see this, one can set $W = M \times [0, 1]$ in Proposition 10.5, and view $M \times \mathbb{R}$ as a manifold obtained from W by attaching cylinders.

Lemma 11.3. *The kernel of the model operator \mathbf{D}^{mod} is G -equivariantly isomorphic (as a graded space) to $\text{Ker}(D_{fv})$. In particular, the index $\chi_G(\mathbf{D}^{\text{mod}})$ is well defined and is equal to $\chi_G(D_{fv})$.*

Proof. Since the operator $c_R(t(x))$ anti-commutes with \tilde{D} and $\tilde{c}(fv)$, we have

$$(\mathbf{D}^{\text{mod}})^2 = \tilde{D}^2 + \left(\sqrt{-1} \gamma \frac{\partial}{\partial t} \otimes c_L(1) + c_R(1) \right)^2.$$

Thus, we obtain the following formulas for the restrictions of $(\mathbf{D}^{\text{mod}})^2$ to the spaces $\Gamma(M \times \mathbb{R}, \mathcal{E}^\pm \otimes \Lambda^\bullet \mathbb{C})$:

$$(\mathbf{D}^{\text{mod}})^2|_{\Gamma(M \times \mathbb{R}, \mathcal{E}^\pm \otimes \Lambda^\bullet \mathbb{C})} = \tilde{D}^2 + \left(\pm \sqrt{-1} \frac{\partial}{\partial t} \otimes c_L(1) + c_R(1) \right)^2. \quad (11.2)$$

Here the first summand coincides with the lift of D_{fv}^2 to $\tilde{\mathcal{E}}$, while the second summand may be considered as an operator acting on the space of $\Lambda^\bullet \mathbb{C}$ -valued functions on \mathbb{R} . Also, both summands in the right hand side of (11.2) are non-negative. Hence, the kernel of $(\mathbf{D}^{\text{mod}})^2$ equals the tensor product of the kernels of these two operators.

The same calculations as in the proof of Lemma 10.4 show that

$$\left(\pm \sqrt{-1} \frac{\partial}{\partial t} \otimes c_L(1) + c_R(1) \right)^2 = -\frac{\partial^2}{\partial t^2} \pm \sqrt{-1} (\Pi_1 - \Pi_0) + t^2$$

The space $\text{Ker} \left(-\frac{\partial^2}{\partial t^2} + \Pi_1 - \Pi_0 + t^2 \right)$ is one dimensional and is spanned by the function $\alpha(t) := e^{-t^2/2} \in \Lambda^0 \mathbb{C}$. Similarly, $\text{Ker} \left(-\frac{\partial^2}{\partial t^2} + \Pi_0 - \Pi_1 + t^2 \right)$ is one dimensional and is spanned by the one-form $\beta(t) := e^{-t^2/2} ds$, where we denote by ds the generator of $\Lambda^1 \mathbb{C}$. It follows that

$$\begin{aligned} \text{Ker}(\mathbf{D}^{\text{mod}})^2|_{\Gamma(M \times \mathbb{R}, \mathcal{E}^+ \otimes \Lambda^\bullet \mathbb{C})} &\simeq \left\{ t^* \sigma \otimes \alpha(t) : \sigma \in \text{Ker} D_{fv}^2|_{\Gamma(M, \mathcal{E}^+)} \right\}; \\ \text{Ker}(\mathbf{D}^{\text{mod}})^2|_{\Gamma(M \times \mathbb{R}, \mathcal{E}^- \otimes \Lambda^\bullet \mathbb{C})} &\simeq \left\{ t^* \sigma \otimes \beta(t) : \sigma \in \text{Ker} D_{fv}^2|_{\Gamma(M, \mathcal{E}^-)} \right\}. \end{aligned}$$

□

11.4. Let $T_a : M \times \mathbb{R} \rightarrow M \times \mathbb{R}$, $T_a(x, t) = (x, t+a)$ be the translation. Using the trivialization of $\tilde{\mathcal{E}}$ along the fibers of $t : M \times \mathbb{R} \rightarrow \mathbb{R}$, we define the pull-back map $T_a^* : \Gamma(M \times \mathbb{R}, \tilde{\mathcal{E}}) \rightarrow \Gamma(M \times \mathbb{R}, \tilde{\mathcal{E}})$. Set

$$\mathbf{D}_a^{\text{mod}} := T_{-a}^* \circ \mathbf{D}^{\text{mod}} \circ T_a^* = \tilde{D} + \sqrt{-1} \tilde{c}(fv) - c_R(t(x) - a)$$

Then $\chi_G(\mathbf{D}_a^{\text{mod}}) = \chi_G(\mathbf{D}^{\text{mod}})$, for all $a \in \mathbb{R}$.

11.5. If A is a self-adjoint operator with discrete spectrum and $\lambda \in \mathbb{R}$, we denote by $N(\lambda, A)$ the number of the eigenvalues of A not exceeding λ (counting multiplicities).

Recall from Subsection 10.6, that $\mathbf{D}_a^{V, \pm}$ denote the restriction of \mathbf{D}_a to the space $\Gamma(\tilde{W}, \tilde{\mathcal{E}}^\pm)^V$. Similarly, let $\mathbf{D}_{V, \pm}^{\text{mod}}, \mathbf{D}_{V, \pm, a}^{\text{mod}}$ denote the restriction of the operators $\mathbf{D}^{\text{mod}}, \mathbf{D}_a^{\text{mod}}$ to the spaces $\Gamma(M \times \mathbb{R}, \tilde{\mathcal{E}}^\pm)^V$.

Proposition 11.6. *Let $\lambda_{V, \pm}$ denote the smallest non-zero eigenvalue of $(\mathbf{D}_{V, \pm}^{\text{mod}})^2$. Then, for every $\varepsilon > 0$, there exists $A = A(\varepsilon, V) > 0$, such that*

$$N(\lambda_{V, \pm} - \varepsilon, (\mathbf{D}_a^{V, \pm})^2) = \dim \text{Ker}(\mathbf{D}_{V, \pm}^{\text{mod}})^2, \quad (11.3)$$

for all $a > A$.

Before proving the proposition let us explain how it implies Theorem 10.8.

11.7. **Proof of Theorem 10.8.** Let V be an irreducible representation of G and let ε and a be as in Proposition 11.6. Let $E_{\varepsilon, a}^{V, \pm} \subset \Gamma(\tilde{W}, \tilde{\mathcal{E}}^\pm)^V$ denote the vector space spanned by the eigenvectors of the operator $(\mathbf{D}_a^{V, \pm})^2$ with eigenvalues smaller or equal to $\lambda_{V, \pm} - \varepsilon$. The operator $\mathbf{D}_a^{V, \pm}$ sends $E_{\varepsilon, a}^{V, \pm}$ into $E_{\varepsilon, a}^{V, \mp}$. Since the dimension of the space $E_{\varepsilon, a}^{V, \pm}$ is finite, it follows that

$$\dim \text{Ker} \mathbf{D}_a^{V, +} - \dim \text{Ker} \mathbf{D}_a^{V, -} = \dim E_{\varepsilon, a}^{V, +} - \dim E_{\varepsilon, a}^{V, -}.$$

By Proposition 11.6, the right hand side of this equality equals $\dim \text{Ker } \mathbf{D}_{V,+}^{\text{mod}} - \dim \text{Ker } \mathbf{D}_{V,+}^{\text{mod}}$. Thus

$$\chi_G(\mathbf{D}_a) = \chi_G(\mathbf{D}^{\text{mod}}).$$

Theorem 10.8 follows now from Lemma 11.3. \square

The rest of this section is occupied with the proof of Proposition 11.6.

11.8. Estimate from above on $N(\lambda_{V,\pm} - \varepsilon, (\mathbf{D}_a^{V,\pm})^2)$. We will first show that

$$N(\lambda_{V,\pm} - \varepsilon, (\mathbf{D}_a^{V,\pm})^2) \leq \dim \text{Ker } \mathbf{D}_{V,\pm}^{\text{mod}}. \quad (11.4)$$

To this end we will estimate the operator \mathbf{D}_a^2 from below. We will use the technique of [17, 5], adding some necessary modifications.

11.9. The IMS localization. Let $j, \bar{j} : \mathbb{R} \rightarrow [0, 1]$ be smooth functions such that $j^2 + \bar{j}^2 \equiv 1$ and $j(t) = 1$ for $t \geq 3$, while $j(t) = 0$ for $t \leq 2$.

Recall that $t : M \times \mathbb{R} \rightarrow \mathbb{R}$ denote the projection and that the map $p : \tilde{W} \rightarrow \mathbb{R}$ was defined in Subsection 10.3. For each $a > 0$, define smooth functions J_a and \bar{J}_a on $M \times \mathbb{R}$ by the formulas:

$$J_a(x) = j(a^{-1/2}t(x)), \quad \bar{J}_a(x) = \bar{j}(a^{-1/2}t(x)).$$

By a slight abuse of notation we will denote by the same letters the smooth functions on \tilde{W} given by the formulas

$$J_a(x) = j(a^{-1/2}p(x)), \quad \bar{J}_a(x) = \bar{j}(a^{-1/2}p(x)).$$

We identify the functions J_a, \bar{J}_a with the corresponding multiplication operators. For operators A, B , we denote by $[A, B] = AB - BA$ their commutator.

The following version of IMS localization formula (cf. [8]) is due to Shubin [18, Lemma 3.1] (see also [5, Lemma 4.10]).

Lemma 11.10. *The following operator identity holds*

$$\mathbf{D}_a^2 = \bar{J}_a \mathbf{D}_a^2 \bar{J}_a + J_a \mathbf{D}_a^2 J_a + \frac{1}{2} [\bar{J}_a, [\bar{J}_a, \mathbf{D}_a^2]] + \frac{1}{2} [J_a, [J_a, \mathbf{D}_a^2]]. \quad (11.5)$$

Proof. Using the equality $J_a^2 + \bar{J}_a^2 = 1$ we can write

$$\mathbf{D}_a^2 = J_a^2 \mathbf{D}_a^2 + \bar{J}_a^2 \mathbf{D}_a^2 = J_a \mathbf{D}_a^2 J_a + \bar{J}_a \mathbf{D}_a^2 \bar{J}_a + J_a [J_a, \mathbf{D}_a^2] + \bar{J}_a [\bar{J}_a, \mathbf{D}_a^2].$$

Similarly,

$$\mathbf{D}_a^2 = \mathbf{D}_a^2 J_a^2 + \mathbf{D}_a^2 \bar{J}_a^2 = J_a \mathbf{D}_a^2 J_a + \bar{J}_a \mathbf{D}_a^2 \bar{J}_a - [J_a, \mathbf{D}_a^2] J_a - [\bar{J}_a, \mathbf{D}_a^2] \bar{J}_a.$$

Summing these identities and dividing by 2, we come to (11.5). \square

We will now estimate each of the summands in the right hand side of (11.5).

Lemma 11.11. *There exists $A > 0$, such that*

$$\bar{J}_a \mathbf{D}_a^2 \bar{J}_a \geq \frac{a^2}{8} \bar{J}_a^2, \quad (11.6)$$

for all $a > A$.

Proof. Note that $p(x) \leq 3a^{1/2}$ for all x in the support of \bar{J}_a . Hence, if $a > 36$, we have $\bar{J}_a^2 |p(x) - a|^2 \geq \frac{a^2}{4} \bar{J}_a^2$.

Recall that the norm $\|B\|_\infty$ was defined in the proof of Proposition 10.5. Set

$$A = \max \{ 36, 4\|B\|_\infty^{1/2} \}$$

and let $a > A$. Using Lemma 10.4, we obtain

$$\bar{J}_a \mathbf{D}_a^2 \bar{J}_a \geq \bar{J}_a^2 |p(x) - a|^2 - \bar{J}_a B \bar{J}_a \geq \frac{a^2}{4} \bar{J}_a^2 - \bar{J}_a^2 \|B\|_\infty \geq \frac{a^2}{8} \bar{J}_a^2.$$

□

11.12. Let $P_a : L_2(M \times \mathbb{R}, \tilde{\mathcal{E}}) \rightarrow \text{Ker } \mathbf{D}_a^{\text{mod}}$ be the orthogonal projection. Let $P_a^{V,\pm}$ denote the restriction of P_a to the space $L_2(M \times \mathbb{R}, \tilde{\mathcal{E}}^\pm)^V$. Then $P_a^{V,\pm}$ is a finite rank operator and its rank equals $\dim \text{Ker } \mathbf{D}_{V,\pm,a}^{\text{mod}}$. Clearly,

$$(\mathbf{D}_{V,\pm,a}^{\text{mod}})^2 + \lambda_{V,\pm} P_a^{V,\pm} \geq \lambda_{V,\pm}. \quad (11.7)$$

By identifying the support of J_a in $M \times \mathbb{R}$ with a subset of \tilde{W} , we can and we will consider $J_a P_a J_a$ and $J_a (\mathbf{D}_a^{\text{mod}})^2 J_a$ as operators on \tilde{W} . Then $J_a \mathbf{D}_a^2 J_a = J_a (\mathbf{D}_a^{\text{mod}})^2 J_a$. Hence, (11.7) implies the following

Lemma 11.13. *For every $a > 0$,*

$$J_a (\mathbf{D}_a^{V,\pm})^2 J_a + \lambda_{V,\pm} J_a P_a^{V,\pm} J_a \geq \lambda_{V,\pm} J_a^2, \quad \text{rk } J_a P_a^{V,\pm} J_a \leq \dim \text{Ker } \mathbf{D}_{V,\pm}^{\text{mod}}. \quad (11.8)$$

For an operator $A : L_2(\tilde{W}, \tilde{\mathcal{E}}) \rightarrow L_2(\tilde{W}, \tilde{\mathcal{E}})$, we denote by $\|A\|$ its norm with respect to L_2 scalar product on $L_2(\tilde{W}, \tilde{\mathcal{E}})$.

Lemma 11.14. *Let $C = 2 \max \{ \max\{|dj(t)|^2, |d\bar{j}(t)|^2\} : t \in \mathbb{R} \}$. Then*

$$\|[J_a, [J_a, \mathbf{D}_a^2]]\| \leq Ca^{-1}, \quad \|[\bar{J}_a, [\bar{J}_a, \mathbf{D}_a^2]]\| \leq Ca^{-1}, \quad \text{for any } a > 0. \quad (11.9)$$

Proof. Since \mathbf{D}_a^2 is a Dirac operator, it follows from [4, Prop. 2.3], that

$$[J_a, [J_a, \mathbf{D}_a^2]] = -2|dJ_a|^2, \quad [\bar{J}_a, [\bar{J}_a, \mathbf{D}_a^2]] = -2|d\bar{J}_a|^2.$$

The lemma follows now from the obvious identities

$$|dJ_a(x)| = a^{-1/2} |dj(a^{-1/2}p(x))|, \quad |d\bar{J}_a(x)| = a^{-1/2} |d\bar{j}(a^{-1/2}p(x))|.$$

□

From Lemmas 11.10, 11.13 and 11.14 we obtain the following

Corollary 11.15. *For every $\varepsilon > 0$, there exists $A = A(\varepsilon, V) > 0$, such that, for all $a > A$, we have*

$$(\mathbf{D}_a^{V,\pm})^2 + \lambda_{V,\pm} J_a P_a^{V,\pm} J_a \geq \lambda_{V,\pm} - \varepsilon, \quad \text{rk } J_a P_a^{V,\pm} J_a \leq \dim \text{Ker } \mathbf{D}_{V,\pm}^{\text{mod}}. \quad (11.10)$$

The estimate (11.4) follows now from Corollary 11.15 and the following general lemma [16, p. 270].

Lemma 11.16. *Assume that A, B are self-adjoint operators in a Hilbert space \mathcal{H} such that $\text{rk } B \leq k$ and there exists $\mu > 0$ such that*

$$\langle (A + B)u, u \rangle \geq \mu \langle u, u \rangle \quad \text{for any } u \in \text{Dom}(A).$$

Then $N(\mu - \varepsilon, A) \leq k$ for all $\varepsilon > 0$.

11.17. Estimate from below on $N(\lambda_{V,\pm} - \varepsilon, (\mathbf{D}_a^{V,\pm})^2)$. To prove Proposition 11.6 it remains now to show that

$$N(\lambda_{V,\pm} - \varepsilon, (\mathbf{D}_a^{V,\pm})^2) \geq \dim \text{Ker } \mathbf{D}_{V,\pm}^{\text{mod}}. \quad (11.11)$$

Let $E_{\varepsilon,a}^{V,\pm} \subset \Gamma(\tilde{W}, \tilde{\mathcal{E}})$ denote the vector space spanned by the eigenvectors of the operator $(\mathbf{D}_a^{V,\pm})^2$ with eigenvalues smaller or equal to $\lambda_{V,\pm} - \varepsilon$. Let $\Pi_{\varepsilon,a}^{V,\pm} : L_2(\tilde{W}, \tilde{\mathcal{E}}^\pm)^V \rightarrow E_{\varepsilon,a}^{V,\pm}$ be the orthogonal projection. Then

$$\text{rk } \Pi_{\varepsilon,a}^{V,\pm} = N(\lambda_{V,\pm} - \varepsilon, (\mathbf{D}_a^{V,\pm})^2). \quad (11.12)$$

As in Subsection 11.12, we can and we will consider $J_a \Pi_{\varepsilon,a}^{V,\pm} J_a$ as an operator on $L_2(M \times \mathbb{R}, \tilde{\mathcal{E}}^\pm)^V$. The proof of the following lemma does not differ from the proof of Corollary 11.15.

Lemma 11.18. *For every $\varepsilon > 0$, there exists $A = A(\varepsilon, V) > 0$, such that, for all $a > A$, we have*

$$\mathbf{D}_{V,\pm,a}^{\text{mod}} + \lambda_{V,\pm} J_a \Pi_a^{V,\pm} J_a \geq \lambda_{V,\pm} - \varepsilon, \quad \text{rk } J_a \Pi_a^{V,\pm} J_a \leq \dim N(\lambda_{V,\pm} - \varepsilon, (\mathbf{D}_a^{V,\pm})^2). \quad (11.13)$$

The estimate (11.11) follows now from (11.12), Lemma 11.18 and Lemma 11.16.

The proof of Proposition 11.6 is complete. \square

12. PROOF OF LEMMA 3.14

12.1. The restriction of the Clifford module to U . Recall that $U_1 = \phi_1(U)$. To simplify the notation we identify U_1 with U and write $U = U_1$. We also denote the boundary ∂U of U in M_1 by Σ . Recall that it is a smooth G -invariant hypersurface in M_1 .

Let $\mathcal{E}_U, \mathbf{v}_U$ denote the restrictions of \mathcal{E}_1 and \mathbf{v}_1 to U , respectively. We will define a structure of a tamed Clifford module on \mathcal{E}_U . For this we need to change the Clifford action of T^*U on \mathcal{E}_U , so that the corresponding Riemannian metric on U is complete.

Let $\alpha : M_1 \rightarrow \mathbb{R}$ be a smooth G -invariant function, such that 0 is a regular value of α and $\alpha^{-1}((0, \infty)) = U$, $\alpha^{-1}(0) = \Sigma$.

Let $c_1 : T^*M_1 \rightarrow \text{End } \mathcal{E}_1$ denote the Clifford module structure on \mathcal{E}_1 . Define a map $c_U : T^*U \rightarrow \text{End } \mathcal{E}_U$ by the formula

$$c_U(a) := \alpha(x)c_1(a), \quad a \in T_x^*U.$$

Then c_U defines a Clifford module structure on \mathcal{E}_U , which corresponds to the Riemannian metric $g^U = \alpha^{-2}g^M|_U$, which is complete. From now on we denote by \mathcal{E}_U the Clifford module defined by c_U . We also endow \mathcal{E}_U with the Hermitian structure obtained by the restriction of the Hermitian structure on \mathcal{E}_1 . Then $(\mathcal{E}_U, \mathbf{v}_U)$ is a tamed Clifford module. Clearly, to prove Lemma 3.14, it is enough to show that this module is cobordant to $(\mathcal{E}_1, \mathbf{v}_1)$.

12.2. Proof of Lemma 3.14. Since we will not work with M_2, \mathcal{E}_2 any more, we will simplify the notation by omitting the subscript “1” everywhere. Thus we set $M = M_1, \mathcal{E} = \mathcal{E}_1$, etc. We will construct now a cobordism between \mathcal{E}_U and \mathcal{E} .

Consider the product $M \times [0, 1]$, and the set

$$Z := \{ (x, t) \in M \times [0, 1] : t \leq 1/3, x \notin U \}.$$

Set $W := (M \times [0, 1]) \setminus Z$. Then W is a non-compact G -manifold, whose boundary is diffeomorphic to the disjoint union of $U \simeq U \times \{0\}$ and $M \simeq M \times \{1\}$. Essentially, W is the required cobordism, but we need to define all the structures on W .

Let $\mu, \nu : W \rightarrow (0, \infty)$ be a smooth G -invariant functions such that

- $\mu(x, t) = 1$, if $t \geq 2/3$;
- $\mu(x, t) = 1/\alpha(x)$, if $t \leq 1/2$ and $\alpha(x) \geq t - 1/3$;

and

- $\nu(x, t) = 1$, if $t \geq 2/3$ or $t \leq 1/4$;
- $\nu(x, t) = 1/t$ if $t \leq 1/2$ and $t - 1/3 \geq \alpha(x)$.

Define the metric g^W on W by the formula

$$g^W \left((\xi_1, \tau_1), (\xi_2, \tau_2) \right) := \mu(x, t)^2 g^M(\xi_1, \xi_2) + \nu(x, t)^2 \tau_1 \tau_2,$$

where $(\xi_1, \tau_1), (\xi_2, \tau_2) \in T_x M \oplus \mathbb{R} \simeq T_{(x,t)} W$. Then g^W is a complete G -invariant metric.

Consider the G -invariant neighborhood

$$\mathcal{O} := \left\{ (x, t) : 4t < \alpha(x) \right\} \sqcup \left\{ (x, t) : x \in M, 3/4 < t \leq 1 \right\} \quad (12.1)$$

of ∂W . Define a map $\phi : (U \times [0, 1/4]) \sqcup (M \times (-1/4, 0]) \rightarrow \mathcal{O}$ by the formulas

$$\begin{aligned} \phi(x, t) &:= (x, t), & x \in U, \quad 0 \leq t < 1/4; \\ \phi(x, t) &:= (x, 1+t), & x \in M, \quad -1/4 < t \leq 0. \end{aligned} \quad (12.2)$$

Clearly, ϕ is a G -equivariant metric preserving diffeomorphism, satisfying condition (iii) of Definition 3.2. Define a map $\mathbf{v}_W : W \rightarrow \mathfrak{g}$ by the formula $\mathbf{v}_W(x, t) = \mathbf{v}(x)$. Then (W, \mathbf{v}_W, ϕ) is a cobordism between (M, \mathbf{v}) and (U, \mathbf{v}_U) .

Let \mathcal{E}_W be the G -equivariant Hermitian vector bundle on W , obtained by restricting to W of the pull-back of \mathcal{E} to $M \times [0, 1]$. Define the map $c_W : T^*W \rightarrow \text{End } \mathcal{E}_W$ by the formula

$$c_W(a, b)e = \mu(x, t)^{-1} c(a)e \pm \nu(x, t)^{-1} b \sqrt{-1} e, \quad e \in \mathcal{E}_{W, (x,t)}^\pm, \quad (a, b) \in T_x^* M \oplus \mathbb{R} \simeq T_{(x,t)}^* W.$$

Then c_W defines a structure of a G -equivariant self-adjoint Clifford module on \mathcal{E}_W , compatible with the Riemannian metric g^W , whose restriction to $U \times \{0\} \subset W$ is isomorphic to \mathcal{E}_U and whose restriction to $M \times \{1\}$ is isomorphic to \mathcal{E} .

One easily checks that the tamed Clifford module $(\mathcal{E}_W, \mathbf{v}_W)$ provides a cobordism between $(\mathcal{E}, \mathbf{v})$ and $(\mathcal{E}_U, \mathbf{v}_U)$. \square

13. PROOF OF THE GLUING FORMULA

13.1. **A cobordism between M and M_Σ .** Consider the product $M \times [0, 1]$, and the set

$$Z := \{ (x, t) \in M \times [0, 1] : t \leq 1/3, x \in \Sigma \}.$$

Set $W := (M \times [0, 1]) \setminus Z$. Then W is an open G -manifold, whose boundary is diffeomorphic to the disjoint union of $M \setminus \Sigma \simeq (M \setminus \Sigma) \times \{0\}$ and $M \simeq M \times \{1\}$. Essentially, W is the required cobordism. However, we have to be accurate in defining a complete Riemannian metric g^W on W , so that the condition (iii) of Definition 3.2 is satisfied.

Recall that the function $\alpha : M \rightarrow [0, 1]$ was defined in Subsection 4.1. Let the function $s : W \rightarrow (0, \infty)$ and the metric g^W on W be as in Subsection 12.2. The group G acts naturally on W preserving the metric g^W . This makes W a complete G -manifold with boundary. Define a G -equivariant map $\mathbf{v}_W : W \rightarrow \mathfrak{g}$ by the formula $\mathbf{v}_W(x, t) = \mathbf{v}(x)$.

We still have some freedom of choosing a Riemannian metric on M_Σ and a Clifford module structure on \mathcal{E}_Σ , cf. Lemma 3.14. To make these choices, consider a map $\varphi : M_\Sigma \rightarrow \partial W$ defined by

$$\varphi(x) = (x, 0).$$

Let g^{M_Σ} be the pull-back to M_Σ of the metric g^W . Then $(M_\Sigma, g^{M_\Sigma}, \mathbf{v}_\Sigma)$ is a tamed G -manifold.

Let \mathcal{O} be a G -invariant neighborhood of ∂W , defined by (12.1). Define a map $\phi : (M_\Sigma \times [0, 1/4]) \sqcup (M \times (-1/4, 0]) \rightarrow \mathcal{O}$ by (12.2). Then ϕ is a G -equivariant metric preserving diffeomorphism, satisfying condition (iii) of Definition 3.2. One easily checks that (W, \mathbf{v}_W, ϕ) is a cobordism between (M, \mathbf{v}) and $(M_\Sigma, \mathbf{v}_\Sigma)$.

13.2. **The bundle \mathcal{E}_W . Proof of Theorem 4.4.** Consider the Clifford module \mathcal{E}_W over W defined as in Subsection 12.2. Then the restriction of \mathcal{E}_W to $M \times \{0\} \subset W$ is isomorphic to \mathcal{E} .

Recall that $\varphi : M_\Sigma \rightarrow W$ is a diffeomorphism of M_Σ onto a piece of boundary of W . Set $\mathcal{E}_{M_\Sigma} = \varphi^* \mathcal{E}_W$. Clearly, \mathcal{E}_{M_Σ} is a G -equivariant \mathbb{Z}_2 -graded self-adjoint Clifford module over the Riemannian manifold (M_Σ, g^{M_Σ}) . Moreover, the restriction of \mathcal{E}_Σ to $\alpha^{-1}(1)$ equals $\mathcal{E}|_{\alpha^{-1}(1)}$.

The tamed Clifford module $(\mathcal{E}_W, \mathbf{v}_W)$ provides a cobordism between $(\mathcal{E}, \mathbf{v})$ and $(\mathcal{E}_\Sigma, \mathbf{v}_\Sigma)$. \square

14. PROOF OF THE INDEX THEOREM

14.1. **A tamed Clifford module over U .** First, we define a complete metric on U and a tamed Clifford module over U , using the construction of Subsection 4.2.

Let $\tau : M \rightarrow \mathbb{R}$ be a smooth G -invariant function such that $\tau^{-1}((0, \infty)) = U$, $\tau^{-1}(0) = \partial U$ and there are no critical values of τ in the interval $[-1, 1]$. Let $r : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function, such that $r(t) = t^2$ for $|t| \leq 1/3$, $r(t) > 1/9$ for $|t| > 1/3$ and $r(t) \equiv 1$ for $|t| > 2/3$. Set $\alpha(x) = r(\tau(x))$. Define a complete G -invariant metric g^U on U by the formula

$$g^U := \frac{1}{\alpha(x)^2} g^M|_U.$$

Define a map $c_U : T^*U \rightarrow \text{End } \mathcal{E}|_U$ by the formula

$$c_U := \alpha(x)c,$$

where $c : T^*M \rightarrow \text{End } \mathcal{E}$ is the Clifford module structure on \mathcal{E} . Then \mathcal{E}_U becomes a G -equivariant \mathbb{Z}_2 -graded self-adjoint Clifford module over U . The pair $(\mathcal{E}_U, \mathbf{v}|_U)$ is a tamed Clifford module. Combining Corollary 4.7 with Lemma 3.12, we obtain

$$\chi_G^{\text{an}}(\mathcal{E}, \mathbf{v}) = \chi_G^{\text{an}}(\mathcal{E}_U, \mathbf{v}|_U). \quad (14.1)$$

Let us fix a Clifford connection $\nabla^{\mathcal{E}_U}$ on \mathcal{E}_U . It follows from the proof of Lemma 2.7 (cf. Section 8), that we can choose an admissible function $f : U \rightarrow [0, \infty)$ for the triple $(\mathcal{E}_U, \mathbf{v}|_U, \nabla^{\mathcal{E}_U})$ so that $f > 1$ and $f(x) \rightarrow \infty$ as $x \rightarrow \infty$. Then the function

$$\bar{f}(x) = \begin{cases} 1/f(x), & x \in U; \\ 0, & x \notin U, \end{cases} \quad (14.2)$$

is continuous.

14.2. A more explicit construction of the topological index. The following explicit construction of $\chi_G^{\text{top}}(\mathcal{E}, \mathbf{v})$ is convenient for our purposes.

Let $\rho : [0, \infty) \rightarrow [0, \infty)$ be a smooth function such that $\rho(t) = 1$ for $t \leq 1$ and $\rho(t) = t$ for $t \geq 2$. Consider a new symbol

$$\sigma'(\xi) := \frac{\sqrt{-1}}{\rho(|\xi|)} (c(\xi) + c(v)), \quad \xi \in T^*M. \quad (14.3)$$

Then σ' is a symbol of order 0.

Let U be as in Subsection 5.4. Then $\sigma'(\xi)$ is invertible for all $\xi \in \pi^{-1}(M \setminus U)$. We now give a more explicit than in Subsection 5.2 construction of the extension of σ' to N .

Fix an open relatively compact subset $\tilde{U} \subset M$ which contains the closure of U . Then there exists a bundle F over \tilde{U} , such that the bundle $\mathcal{E}^+|_{\tilde{U}} \oplus F$ is trivial. Consider the symbol

$$\tilde{\sigma}' := \sigma'|_{\tilde{U}} \oplus \text{Id} \in \Gamma(\tilde{U}, \text{Hom}(\mathcal{E}^+|_{\tilde{U}} \oplus F, \mathcal{E}^-|_{\tilde{U}} \oplus F)).$$

The map $c(v) \oplus \text{Id}$ defines an isomorphism between the restrictions of $\mathcal{E}^+|_{\tilde{U}} \oplus F$ and $\mathcal{E}^-|_{\tilde{U}} \oplus F$ to $\tilde{U} \setminus U$, and, hence, a trivialization of $\mathcal{E}^-|_{\tilde{U}} \oplus F$ over $\tilde{U} \setminus U$.

Let $j : \tilde{U} \rightarrow N$ be a G -equivariant embedding of \tilde{U} into a compact G -manifold N . Then the bundles $\mathcal{E}^\pm|_{\tilde{U}} \oplus F$ extend naturally to bundles $\tilde{\mathcal{E}}_N^\pm$ over N and the symbol $\tilde{\sigma}'$ extends naturally to a zeroth-order transversally elliptic symbol $\tilde{\sigma}'_N$ on N , whose restriction to $N \setminus U$ is the identity map.

The symbol $\tilde{\sigma}'_N$ is homotopic to the symbol $\tilde{\sigma}_N$ of Subsection 5.3. Hence, these 2 symbols have the same indexes and we obtain

$$\chi_G^{\text{top}}(\sigma) := \chi_G(\tilde{\sigma}'_N). \quad (14.4)$$

14.3. A homotopy of the symbol $\tilde{\sigma}'_N$. Let $\bar{c} : \tilde{\mathcal{E}}^+ \rightarrow \tilde{\mathcal{E}}^-$ denote the map, whose restriction to \tilde{U} is $c(v) \oplus \text{Id}$ and whose restriction to $N \setminus U$ is the identity map. Recall that the function \bar{f} was defined in the end of Subsection 14.1. Set

$$\hat{\sigma}_N(\xi) = \sqrt{-1} \bar{c} + \frac{\sqrt{-1} \bar{f}}{\rho(|\xi|)} c(\xi), \quad \xi \in T^*N.$$

Clearly, $\hat{\sigma}_N$ is homotopic to $\hat{\sigma}'_N$.

14.4. A transversally elliptic operator with symbol $\hat{\sigma}_N$. We now construct a particular zero-order transversally elliptic operator P on N , whose symbol is equal to $\hat{\sigma}_N$ and, consequently, whose index is equal to $\chi_G^{\text{top}}(\mathcal{E}, \mathbf{v})$.

Let $A : \Gamma(N, \tilde{E}^+) \rightarrow \Gamma(N, \tilde{E}^+)$ be an invertible positive-definite self-adjoint G -invariant second-order differential operator, whose symbol is equal to $|\xi|^2$.

Let $D_U^\pm : \Gamma(U, \mathcal{E}_U^\pm) \rightarrow \Gamma(U, \mathcal{E}_U^\mp)$ be the Dirac operator associated to the Clifford connection $\nabla^{\mathcal{E}_U}$, cf. Subsection 14.1. Since $\text{supp } \bar{f}$ coincides with the closure of U , we can and we will consider the product $\bar{f}D_U$ as an operator on N .

Set

$$P = \sqrt{-1}\bar{c} + \bar{f}\alpha^{-1}D_U^+A^{-1/2}.$$

Then the symbol of P is equal to $\hat{\sigma}_N$ ⁷. Hence,

$$\chi_G^{\text{top}}(\mathcal{E}, \mathbf{v}) = \chi_G(P). \quad (14.5)$$

14.5. A deformation of P . Consider the family of operators

$$P_t = (1-t)\sqrt{-1}\bar{c} + t\sqrt{-1}\bar{c}A^{-1/2} + \bar{f}\alpha^{-1}D_U^+A^{-1/2}, \quad t \in [0, 1].$$

Then $P_0 = P$.

For every irreducible representation $V \in \text{Irr } G$, let us denote by P_t^V , ($t \in [0, 1]$) the restriction of P_t to the isotypical component corresponding to V .

For each t_1, t_2 , the difference $P_{t_1} - P_{t_2}$ is a bounded operator, depending continuously on t_1 and t_2 . Also, for all $t < 1$, the operator P_t is transversally elliptic. Therefore, for every $V \in \text{Irr } G$ and every $t < 1$, the operator P_t^V is Fredholm. Hence, $\chi_G(P_t) = \chi_G(P_0)$ for every $t < 1$. Moreover, to show that $\chi_G(P_1) = \chi_G(P_0)$ we only need to prove that the operator P_1^V is Fredholm for all $V \in \text{Irr } G$.

14.6. The operator P_1 . Proof of Theorem 5.5. Let us investigate $\text{Ker } P_1$. Note, first, that

$$P_1 = \sqrt{-1}\bar{c}A^{-1/2} + \bar{f}\alpha^{-1}D_U^+A^{-1/2}.$$

Hence, $u \in \text{Ker } P_1$ if and only if $w := A^{-1/2}u$ satisfy

$$(\sqrt{-1}\bar{c} + \bar{f}\alpha^{-1}D_U^+)w = 0. \quad (14.6)$$

Since, $\bar{f} \equiv 0$ and $\bar{c} \equiv \text{Id}$ on $N \setminus U$, it follows from (14.6), that $w \equiv 0$ on $N \setminus U$. Hence, (14.6) is satisfied if and only if $\text{supp } w$ lies in the closure of U and (14.6) holds on U . Recall that on U we have $\bar{f} = 1/f$, $\bar{c} = c(v)$. Hence, (14.6) is equivalent to

$$\left(\sqrt{-1}c(v) + \frac{1}{f\alpha}D_U^+ \right)w = 0 \Leftrightarrow (D_U^+ + \sqrt{-1}f\alpha c(v))w = 0.$$

⁷Note, that the symbol of the operator $\alpha^{-1}D_U^+$ is equal to $-\sqrt{-1}c(\xi)$. Therefore, though the function α^{-1} tends to infinity near the boundary of U , the coefficients of the differential operator $\bar{f}\alpha^{-1}D_U^+$ are continuous in any coordinate chart. Hence, the pseudo-differential operator P is well defined.

Since, $\alpha c(v) = c_U(v)$, the later equation is equivalent to $(D_U^+ + \sqrt{-1}fc_U(v))w = 0$. Since, $A^{-1/2}$ is invertible we see that $\text{Ker } P_1$ is equivariantly isomorphic to $\text{Ker}(D_U^+ + \sqrt{-1}fc_U(v))$. Similarly, one shows that $\text{Coker } P_1$ is equivariantly isomorphic to $\text{Ker}(D_U^- + \sqrt{-1}fc_U(v))$. Therefore

$$\chi_G(P_1) = \chi_G(D_U + \sqrt{-1}fc_U(v)) := \chi_G^{\text{an}}(\mathcal{E}|_U, \mathbf{v}|_U). \quad (14.7)$$

In particular, we see that P_1^V is Fredholm for every $V \in \text{Irr } G$. Hence, as it was explained in the end of the previous subsection,

$$\chi_G(P_1) = \chi_G(P).$$

Theorem 5.5 follows now from (14.1), (14.5) and (14.7). \square

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