

# WITTEN DEFORMATION OF THE ANALYTIC TORSION AND THE SPECTRAL SEQUENCE OF A FILTRATION

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ABSTRACT. Let  $F$  be a flat vector bundle over a compact Riemannian manifold  $M$  and let  $f : M \rightarrow \mathbb{R}$  be a Morse function. Let  $g^F$  be a smooth Euclidean metric on  $F$ , let  $g_t^F = e^{-2tf}g^F$  and let  $\rho^{RS}(t)$  be the Ray-Singer analytic torsion of  $F$  associated to the metric  $g_t^F$ . Assuming that  $\nabla f$  satisfies the Morse-Smale transversality conditions, we provide an asymptotic expansion for  $\log \rho^{RS}(t)$  for  $t \rightarrow +\infty$  of the form  $a_0 + a_1 t + b \log(\frac{t}{\pi}) + o(1)$ , where the coefficient  $b$  is a half-integer depending only on the Betti numbers of  $F$ . In the case where all the critical values of  $f$  are rational, we calculate the coefficients  $a_0$  and  $a_1$  explicitly in terms of the spectral sequence of a filtration associated to the Morse function. These results are obtained as an applications of a theorem by Bismut and Zhang.

## 0. INTRODUCTION

0.1. The *analytic torsion*,  $\rho^{RS}$ , introduced by Ray and Singer [RS], is a numerical invariant associated to a flat Euclidean vector bundle  $F$  over a compact Riemannian manifold  $M$ . It depends smoothly on the Riemannian metric  $g^{TM}$  on  $TM$  and on the Euclidean metric  $g^F$  on  $F$ .

Let  $f : M \rightarrow \mathbb{R}$  be a Morse function. By Witten deformation of the Euclidean bundle  $F$  we shall understand the family of metrics

$$g_t^F = e^{-2tf}g^F, \quad t > 0 \tag{0.1}$$

on  $F$ . Let  $\rho^{RS}(t)$  be the Ray-Singer torsion of  $F$  associated to the metrics  $g^{TM}$  and  $g_t^F$ .

Burghlea, Friedlander and Kappeler ([BFK3]) have shown (with some additional conditions on  $f$ , cf. Section 0.2) that the function

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$\log \rho^{RS}(t)$  has an asymptotic expansion for  $t \rightarrow +\infty$  of the form

$$\log \rho^{RS}(t) = \sum_{j=0}^{n+1} a_j t^j + b \log t + o(1). \quad (0.2)$$

Suppose that the dimension of  $M$  is odd and that the Morse function is self-indexing (i.e.  $f(x) = \text{index}(x)$  for any critical point  $x$  of  $f$ ). Theorem A of [BFK3] implies that in this case (0.2) reduces to the expansion

$$\log \rho^{RS}(t) = a_0 + a_1 t + b \log \left( \frac{t}{\pi} \right) + o(1) \quad (0.3)$$

and also provides formulae for the coefficients  $a_0, a_1, b$ .

In the present paper we apply the Bismut-Zhang theorem about the comparison of analytic and combinatorial torsion ([BZ1, Theorem 0.2]) to prove that (0.3) remains true in the general case, when  $\dim M$  is not necessarily odd and the Morse function is not necessarily self-indexing.

In the case where all the critical values of  $f$  are rational we calculate explicitly the coefficients of the asymptotic expansion (0.3) in terms of the spectral sequence of a filtration associated to  $f$ . These calculations are based on the descriptions of the singularities of the torsion obtained by Farber [Fa].

Now we shall discuss the precise formulations of these results.

**0.2. Assumptions on  $f, g^F$  and  $g^{TM}$ .** Let  $f : M \rightarrow \mathbb{R}$  be a Morse function. Denote by  $\nabla f$  the gradient vector field of  $f$  with respect to the metric  $g^{TM}$ . Let  $B$  be the finite set of zeroes of  $\nabla f$ .

We shall assume that the following conditions are satisfied (cf. [BFK3, page 5]):

- (1) The gradient vector field  $\nabla f$  satisfies *the Smale transversality conditions* [Sm1, Sm2] (for any two critical points  $x$  and  $y$  of  $f$  the stable manifold  $W^s(x)$  and the unstable manifold  $W^u(y)$ , with respect to  $\nabla f$ , intersect transversally).
- (2) For any  $x \in B$ , the metric  $g^F$  is flat near  $B$  and there is a system of coordinates  $y = (y^1, \dots, y^n)$  centered at  $x$  such that near  $x$

$$g^{TM} = \sum_{i=1}^n |dy^i|^2, \quad f(y) = f(x) - \frac{1}{2} \sum_{i=1}^{\text{index}(x)} |y^i|^2 + \frac{1}{2} \sum_{i=\text{index}(x)+1}^n |y^i|^2.$$

**0.3. The Milnor torsion.** The *Thom-Smale complex* is a finite dimensional complex generated by the fibers of  $F_x$  ( $x \in B$ ) of  $F$  whose cohomology is canonically isomorphic to  $H^\bullet(M, F)$ . The metric  $g^F$  on

$F$  defines the Euclidean structure on the Thom-Smale complex. The Witten deformation  $g_t^F = e^{-2tf}g^F$  of the metric on  $F$  determines the deformation of this structure. We shall refer to this deformation as to the Witten deformation of the Thom-Smale complex.

The *Milnor torsion*  $\rho^{\mathcal{M}}(t)$  is the torsion of the Thom-Smale complex corresponding to the metric  $g_t^F$  on  $F$ .

**Lemma 0.4.** *The function  $\rho^{\mathcal{M}}(t)$  admits an asymptotic expansion as  $t \rightarrow +\infty$  of the form*

$$\log \rho^{\mathcal{M}}(t) = \alpha + \beta t + o(1). \quad (0.4)$$

Lemma 0.4 is proved in Section 2.9.

**0.5. The case where all the critical values of  $f$  are rational.** Suppose now all the critical values  $f_1 < \dots < f_l$  of  $f$  are rational. In this situation we shall calculate the coefficients  $\alpha, \beta$  of the asymptotic expansion (0.4).

Assume that  $d \in \mathbb{N}$  and  $p_1, \dots, p_l \in \mathbb{Z}$  are such that

$$f_i = \frac{p_i}{d}, \quad (1 \leq i \leq l). \quad (0.5)$$

After the change of the deformation parameter  $t \mapsto \tau = e^{-\frac{t}{d}}$  the Witten deformation of the Thom-Smale complex satisfies the conditions of Theorem 6.6 of [Fa]. Hence, the coefficients  $\alpha, \beta$  of the asymptotic expansion (0.4) may be calculated in terms of the spectral sequence of this deformation (cf. Section 3.2). This spectral sequence admits the following geometric description.

Let  $m, k \in \mathbb{Z}$  be such that  $(m - k)/d < f(x) < m/d$  for any  $x \in M$ . We define the filtration  $\emptyset = U^0 \dots U^k = M$  on  $M$  by

$$U^i = f^{-1} \left[ \frac{m - i}{d}, \frac{m}{d} \right] \quad (0 \leq i \leq k). \quad (0.6)$$

In Section 5 we show that the spectral sequence of the Witten deformation of the Thom-Smale complex may be expressed in terms of the spectral sequence  $(E_r^{p,q}, d_r)$  associated with this filtration.

*Remark 0.6.* The filtration (0.6) and the spectral sequence  $(E_r^{p,q}, d_r)$  depend on the choice of  $d$  in (0.5). Unfortunately, the spectral sequence associated with seemingly more natural filtration  $\emptyset = V^0 \dots V^l = M$ ,  $V^i = f^{-1}[f_i, m/d]$  is not connected to the spectral sequence of the Witten deformation.

Let  $\rho^{ss}$  be the torsion of the spectral sequence  $(E_r^{p,q}, d_r)$  (cf. Definition 4.14). Note that our definition of the torsion of a spectral sequence is slightly different from [Fa] (cf. Remark 4.15).

**Proposition 0.7.** *If all the critical values of  $f$  are rational, then the coefficients  $\alpha$  and  $\beta$  in (0.4) are given by the formulae*

$$\alpha = \log \rho^{ss}; \quad (0.7)$$

$$\beta = -\frac{1}{d} \left( \sum_{p,q \geq 0} (-1)^{p+q} (p+q) \sum_{r \geq 1} r (\dim E_r^{p,q} - \dim E_{r+1}^{p,q}) \right). \quad (0.8)$$

Proposition 0.7 is proved in Section 6.

*Remark 0.8.* The assumption that the critical values of  $f$  are rational does not seem natural. It would be very interesting to obtain formulae for the coefficients  $\alpha, \beta$  of (0.4) without this assumption. These formulae would represent  $\alpha, \beta$  as functions of the critical values of the Morse function  $f$ . Unfortunately, those functions are not continuous. One can show that they can have jumps when the numbers  $f_1, \dots, f_l$  are rationally dependent.

Note that, if the critical points of  $f$  are not rational, the substitution  $t \mapsto \tau = e^{-\frac{t}{d}}$  is not defined and, hence, Farber's theorem can not be applied to the study of the Witten deformation of the Thom-Smale complex.

**0.9. Notation.** Following [BZ1], we introduce the following definitions.

Let  $\nabla^{TM}$  be the Levi-Civita connection on  $TM$  corresponding to the metric  $g^{TM}$ , and let  $e(TM, \nabla^{TM})$  be the associated representative of the Euler class of  $TM$  in Chern-Weil theory.

Let  $\psi(TM, \nabla^{TM})$  be the Mathai-Quillen ([MQ, §7])  $n-1$  current on  $TM$  (see also [BGS, Section 3] and [BZ1, Section III d]) which restriction on  $TM/\{0\}$  is induced by a smooth form on the sphere bundle which transgresses the form  $e(TM, \nabla^{TM})$ .

Let  $\nabla^F$  denote the flat connection on  $F$  and let  $\theta(F, g^F)$  be the 1-form on  $M$  defined by (cf. [BZ1, Section IV d])

$$\theta(F, g^F) = \text{Tr} [(g^F)^{-1} \nabla^F g^F]. \quad (0.9)$$

Set

$$\chi(F) = \sum_{i=0}^n (-1)^i \dim H^i(M, F), \quad (0.10)$$

$$\chi'(F) = \sum_{i=0}^n (-1)^i i \dim H^i(M, F), \quad (0.11)$$

$$\text{Tr}_s^B [f] = \sum_{x \in B} (-1)^{\text{index}(x)} f(x). \quad (0.12)$$

Our main result is the following

**Theorem 0.10.** *Let  $\rho^{RS}(t)$  be the analytic torsion corresponding to the metric  $g_t^F = e^{-2tf}g^F$ .*

(i) *The function  $\log \rho^{RS}(t)$  admits an asymptotic expansion for  $t \rightarrow +\infty$  of the form*

$$\log \rho^{RS}(t) = a_0 + a_1 t + b \log \left( \frac{t}{\pi} \right) + o(1), \quad (0.13)$$

where

$$b = \frac{n}{4}\chi(F) - \frac{1}{2}\chi'(F) \quad (0.14)$$

and  $a_0, a_1$  are real numbers depending on  $f$ ,  $g^F$  and  $g^{TM}$ .

(ii) *Assume that all the critical values of  $f$  are rational and let the integer  $d$  and the spectral sequence  $(E_r^{p,q}, d_r)$  be as in Section 0.5. Then the coefficients  $a_0$  and  $a_1$  in (0.13) are given by the formulae*

$$a_0 = \log \rho^{ss} - \frac{1}{2} \int_M \theta(F, g^F)(\nabla f)^* \psi(TM, \nabla^{TM}); \quad (0.15)$$

$$\begin{aligned} a_1 = & -\text{rk}(F) \int_M f e(TM, \nabla^{TM}) \\ & - \frac{1}{d} \sum_{p,q \geq 0} (-1)^{p+q} (p+q) \sum_{r \geq 1} r (\dim E_r^{p,q} - \dim E_{r+1}^{p,q}) + \text{rk}(F) \text{Tr}_s^B[f]. \end{aligned} \quad (0.16)$$

*Remark 0.11.* The Witten deformation is a deformation of an elliptic complex. If this deformation were elliptic with parameter ([Sh], [BFK1]) then, by [BFK1, Theorem A.3], its torsion would have an asymptotic expansion for  $t \rightarrow \infty$ , whose coefficients would be given by local expressions. It fails to be elliptic with parameter precisely at the critical points of the Morse function  $f$ . In [BFK3], Burghelea, Friedlander and Kappeler used the Mayer-Vietoris formula for elliptic operators ([BFK1]) to show that its torsion continues to have an asymptotic expansion with coefficients which are not local anymore. Theorem 0.10 provides explicit formulae for these coefficients.

*Remark 0.12.* The connection between the asymptotic behavior of the analytic torsion and the spectral sequence associated with the deformation was discovered by Farber [Fa]. We discuss Farber's results in Section 4.

In [DM], Dai and Melrose have obtained the asymptotic of the Ray-Singer analytic torsion in the adiabatic limit. Their result is also expressed in terms of a spectral sequence.

**0.13. The case where the Morse function is self-indexing.** Suppose now that the Morse function  $f : M \rightarrow \mathbb{R}$  is self-indexing and choose  $d = 1$  (cf. Section 0.5). For  $0 \leq i \leq n$ , let  $m_i$  denote the number of  $x \in B$  of index  $i$ . Then (cf. [BZ1, page 30]) the spectral sequence  $(E_r^{p,q}, d_r)$  degenerates in the second term and

$$\begin{aligned} \dim E_1^{p,q} &= \begin{cases} m_p \operatorname{rk}(F), & \text{if } q = 0; \\ 0, & \text{if } q \neq 0. \end{cases} \\ \dim E_2^{p,q} &= \begin{cases} \dim H^p(M, F), & \text{if } q = 0; \\ 0, & \text{if } q \neq 0. \end{cases} \end{aligned} \quad (0.17)$$

Hence,

$$\sum_{p,q \geq 0} (-1)^{p+q} (p+q) \sum_{r \geq 1} r (\dim E_r^{p,q} - \dim E_{r+1}^{p,q}) = \operatorname{rk}(F) \operatorname{Tr}_s^B[f] - \chi'(F). \quad (0.18)$$

Also, the torsion  $\rho^{ss}$  of the spectral sequence is easily seen to be equal to the Milnor torsion  $\rho^{\mathcal{M}}$ .

We obtain the following corollary (cf. [BFK3, Theorem A], [Br, Theorem 0.4])

**Corollary 0.14.** *If the Morse function  $f : M \rightarrow \mathbb{R}$  is self-indexing, then the coefficients  $a_0, a_1$  of the asymptotic expansion (0.13) are given by the formulae*

$$a_0 = \log \rho^{\mathcal{M}} - \frac{1}{2} \int_M \theta(F, g^F) (\nabla f)^* \psi(TM, \nabla^{TM}); \quad (0.19)$$

$$a_1 = -\operatorname{rk}(F) \int_M fe(TM, \nabla^{TM}) + \chi'(F). \quad (0.20)$$

**0.15. The method of the proof.** Our method is completely different from that of [BFK3]. In [BFK3] the asymptotic expansion is proved by direct analytic arguments and, then is applied to get a new proof of the Ray-Singer conjecture [RS] (which was originally proved by Cheeger [Ch] and Müller [Mü1]).

In Section 8 of the present paper, we use the Bismut-Zhang extension of this conjecture ([BZ1, Theorem 0.2]) to get the following proposition

**Proposition 0.16.** *As  $t \rightarrow +\infty$ , the following identity holds*

$$\begin{aligned} \log \rho^{RS}(t) - \log \rho^{\mathcal{M}}(t) &= \\ &= -\frac{1}{2} \int_M \theta(F, g^F) (\nabla f)^* \psi(TM, \nabla^{TM}) - t \operatorname{rk}(F) \int_M fe(TM, \nabla^{TM}) \\ &\quad + t \operatorname{rk}(F) \operatorname{Tr}_s^B[f] + \left( \frac{n}{4} \chi(F) - \frac{1}{2} \chi'(F) \right) \log \left( \frac{t}{\pi} \right) + o(1). \end{aligned} \quad (0.21)$$

Theorem 0.10 follows from Proposition 0.16, Lemma 0.4 and Proposition 0.7.

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## 1. THE TORSION OF A FINITE DIMENSIONAL COMPLEX

In this section we follow [BZ1, Section 1a].

**1.1. The determinant line.** If  $\lambda$  is a real line, let  $\lambda^{-1}$  be the dual line. If  $E$  is a finite dimensional vector space, set

$$\det E = \bigwedge^{\max} (E)$$

Let

$$(V^\bullet, \partial) : 0 \rightarrow V^0 \xrightarrow{\partial} \dots \xrightarrow{\partial} V^n \rightarrow 0 \quad (1.1)$$

be a complex of finite dimensional Euclidean vector spaces. Let  $H^\bullet(V) = \bigoplus_{i=0}^n H^i(V)$  be the cohomology of  $(V^\bullet, \partial)$ . Set

$$\det V^\bullet = \bigotimes_{i=0}^n (\det V^i)^{(-1)^i}, \quad (1.2)$$

$$\det H^\bullet(V) = \bigotimes_{i=0}^n (\det H^i(V))^{(-1)^i}. \quad (1.3)$$

Then, by [KM $\mu$ ], there is a canonical isomorphism of real lines

$$\det H^\bullet(V) \simeq \det V^\bullet. \quad (1.4)$$

**1.2. Two metrics on the determinant line.** The Euclidean structure on  $V^\bullet$  defines a metric on  $\det V^\bullet$ . Let  $\|\cdot\|_{\det H^\bullet(V)}$  be the metric on the line  $\det H^\bullet(V)$  corresponding to this metric via the canonical isomorphism (1.4).

Let  $\partial^*$  be the adjoint of  $\partial$  with respect to the Euclidean structure on  $C^\bullet(W^u, F)$ . Using the finite dimensional Hodge theory, we have the canonical identification

$$H^i(V^\bullet, \partial) \simeq \{v \in V^i : \partial v = 0, \partial^* v = 0\}, \quad 0 \leq i \leq n. \quad (1.5)$$

As a vector subspace of  $V^i$ , the vector space in the right-hand side of (1.5) inherits the Euclidean metric. We denote by  $|\cdot|_{\det H^\bullet(V)}$  the corresponding metric on  $\det H^\bullet(V)$ . We shall refer to this metric as to the *Hodge metric* on  $\det H^\bullet(V)$ .

The metrics  $\|\cdot\|_{\det H^\bullet(V)}$  and  $|\cdot|_{\det H^\bullet(V)}$  do not coincide in general. We shall describe the discrepancy.

**1.3. The torsion of a finite dimensional complex.** Set  $\Delta = \partial\partial^* + \partial^*\partial$  and let  $\Pi : V^\bullet \rightarrow \text{Ker } \Delta$  be the orthogonal projection. Set  $\Pi^\perp = 1 - \Pi$ .

Let  $N$  and  $\tau$  be the operators on  $V^\bullet$  acting on  $V^i$  ( $0 \leq i \leq n$ ) by multiplication by  $i$  and  $(-1)^i$  respectively. If  $A \in \text{End}(V^\bullet)$ , we define the supertrace  $\text{Tr}_s[A]$  by the formula

$$\text{Tr}_s[A] = \text{Tr}[\tau A]. \quad (1.6)$$

For  $s \in \mathbb{C}$ , set

$$\zeta^V(s) = -\text{Tr}_s [N(\Delta)^{-s}\Pi^\perp].$$

**Definition 1.4.** The *torsion* of the complex  $(V^\bullet, d)$  is the number

$$\rho = \exp\left(\frac{1}{2} \frac{d\zeta^V(0)}{ds}\right). \quad (1.7)$$

We denote by  $\Delta^i$  ( $0 \leq i \leq n$ ) the restriction of  $\Delta$  on  $V^i$ . Let  $\{\lambda_j^i\}$  be the set of nonzero eigenvalues of  $\Delta^i$ . Then

$$\log \rho = \frac{1}{2} \sum_{i,j} (-1)^i i \log \lambda_j^i. \quad (1.8)$$

The following result is proved in [BGS, Proposition 1.5]

$$\|\cdot\|_{\det H^\bullet(V)} = |\cdot|_{\det H^\bullet(V)} \cdot \rho. \quad (1.9)$$

## 2. THE MILNOR METRIC AND THE MILNOR TORSION

In this section we recall the definitions of the Milnor metric and the Milnor torsion and prove Lemma 0.4.

**2.1. The determinant line of the cohomology.** Let  $H^\bullet(M, F) = \bigoplus_{i=0}^n H^i(M, F)$  be the cohomology of  $M$  with coefficients in  $F$  and let  $\det H^\bullet(M, F)$  be the line

$$\det H^\bullet(M, F) = \bigotimes_{i=0}^n \left( \det H^i(M, F) \right)^{(-1)^i}. \quad (2.1)$$

**2.2. The Thom-Smale complex.** Let  $f : M \rightarrow \mathbb{R}$  be a Morse function satisfying the Smale transversality conditions [Sm1, Sm2] (for any two critical points  $x$  and  $y$  of  $f$  the stable manifold  $W^s(x)$  and the unstable manifold  $W^u(y)$ , with respect to  $\nabla f$ , intersect transversally).

Let  $B$  be the set of critical points of  $f$ . If  $x \in B$ , let  $F_x$  denote the fiber of  $F$  over  $x$  and let  $[W^u(x)]$  denote the real line generated by  $W^u(x)$ . For  $0 \leq i \leq n$ , set

$$C^i(W^u, F) = \bigoplus_{\substack{x \in B \\ \text{index}(x)=i}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x. \quad (2.2)$$

By a basic result of Thom ([Th]) and Smale ([Sm2]) (see also [BZ1, pages 28–30]), there are well defined linear operators

$$\partial : C^i(W^u, F) \rightarrow C^{i+1}(W^u, F),$$

such that the pair  $(C^\bullet(W^u, F), \partial)$  is a complex and there is a canonical identification of  $\mathbb{Z}$ -graded vector spaces

$$H^\bullet(C^\bullet(W^u, F), \partial) \simeq H^\bullet(M, F). \quad (2.3)$$

**2.3. The Milnor metric.** By (1.4) and (2.3), we know that

$$\det H^\bullet(M, F) \simeq \det C^\bullet(W^u, F). \quad (2.4)$$

The metric  $g^F$  on  $F$  determines the structure of an Euclidean vector space on  $C^\bullet(W^u, F)$ . This structure induces a metric on  $\det C^\bullet(W^u, F)$ .

**Definition 2.4.** The *Milnor metric*  $\|\cdot\|_{\det H^\bullet(M, F)}^{\mathcal{M}}$  on the line  $\det H^\bullet(M, F)$  (cf. [BZ1, Section Id]) is the metric corresponding to the above metric on  $\det C^\bullet(W^u, F)$  via the canonical isomorphism (2.4).

*Remark 2.5.* By Milnor [Mi1, Theorem 9.3], if  $g^F$  is a flat metric on  $F$ , then the Milnor metric coincides with the Reidemeister metric defined through a smooth triangulation of  $M$ . In this case  $\|\cdot\|_{\det H^\bullet(M, F)}^{\mathcal{M}}$  does not depend upon  $f$  and  $g^{TM}$  and, hence, is a topological invariant of the flat Euclidean vector bundle  $F$ .

**Definition 2.6.** The *Hodge-Milnor metric*  $|\cdot|_{\det H^\bullet(M, F)}^{\mathcal{M}}$  on  $\det H^\bullet(M, F)$  is the metric corresponding to the Hodge metric (cf. Section 1.2) on  $\det H^\bullet(C^\bullet(W^u, F), \partial)$  via (2.4).

**Definition 2.7.** The *Milnor torsion* is the torsion of the Thom-Smale complex (cf. Definition 1.4).

From (1.9), we obtain

$$\|\cdot\|_{\det H^\bullet(M, F)}^{\mathcal{M}} = |\cdot|_{\det H^\bullet(M, F)}^{\mathcal{M}} \cdot \rho^{\mathcal{M}}. \quad (2.5)$$

**2.8. Deformation of the Milnor metric.** The metric  $\|\cdot\|_{\det H^\bullet(M,F)}^{\mathcal{M}}$  depends on the metric  $g^F$ . Let  $g_t^F = e^{-2tf}g^F$  and let  $\|\cdot\|_{\det H^\bullet(M,F),t}^{\mathcal{M}}$  be the corresponding Milnor metric. Recall from Section 0.9 the notation

$$\mathrm{Tr}_s^B[f] = \sum_{x \in B} (-1)^{\mathrm{index}(x)} f(x). \quad (2.6)$$

Obviously,

$$\|\cdot\|_{\det H^\bullet(M,F),t}^{\mathcal{M}} = e^{-t \mathrm{rk}(F) \mathrm{Tr}_s^B[f]} \cdot \|\cdot\|_{\det H^\bullet(M,F)}^{\mathcal{M}}. \quad (2.7)$$

**2.9. Proof of Lemma 0.4.** Assume that  $g_t$  is the metric on  $C^\bullet(W^u, F)$  induced by the metric  $g_t^F = e^{-2tf}g^F$  on  $F$ . Let  $\mathcal{F} \in \mathrm{End}(C^\bullet(W^u, F))$  which, for  $x \in B$ , acts on  $[W^u(x)]^* \otimes F_x$  by multiplication by  $f(x)$ . Then, for any  $x, y \in C^\bullet(W^u, F)$ , we have

$$g_t(x, y) = g_0(e^{-2t\mathcal{F}}x, y). \quad (2.8)$$

Let  $\partial_t^*$  be the adjoint of  $\partial$  with respect to the metric  $g_t$ . Clearly,

$$\partial_t^* = e^{2t\mathcal{F}}\partial_0^*e^{-2t\mathcal{F}}. \quad (2.9)$$

Set  $\Delta_t = \partial\partial_t^* + \partial_t^*\partial$  and denote by  $\Delta_t^i$  the restriction of  $\Delta_t$  on  $C^i(W^u, F)$ . The number  $k^i = \dim \mathrm{Ker} \Delta_t^i$  does not depend on  $t$ . Hence, the characteristic polynomial  $\det(\Delta_t^i - xI)$  of  $\Delta_t^i$  may be written in the form

$$\det(\Delta_t^i - xI) = x^{k^i} \sum_{j=0}^{\dim V^i - k^i} a_j^i(t)x^j, \quad (2.10)$$

where  $a_0^i(t) \neq 0$  is equal to the product of the nonzero eigenvalues of  $\Delta_t^i$ .

Let  $\rho^{\mathcal{M}}(t)$  denote the Milnor torsion corresponding to the metric  $g_t^F$ . From (1.8), we see that

$$\log \rho^{\mathcal{M}}(t) = \frac{1}{2} \sum_{i=0}^n (-1)^i i a_0^i(t). \quad (2.11)$$

By (2.9),  $a_0^i(t)$  is a polynomial in  $e^{-2tf_1}, \dots, e^{-2tf_l}$ . Hence, for any  $0 \leq i \leq n$ , there exist real numbers  $\alpha^i, \beta^i$  such that, for  $t \rightarrow +\infty$ ,

$$a_0^i(t) = \alpha^i e^{\beta^i t} (1 + o(1)). \quad (2.12)$$

From (2.11) and (2.12), we obtain

$$\log \rho^{\mathcal{M}}(t) = \frac{1}{2} \sum_{i=0}^n (-1)^i i \log \alpha^i + \frac{1}{2} t \sum_{i=0}^n (-1)^i i \beta^i + o(1), \quad (2.13)$$

completing the proof of the lemma.

## 3. THE SPECTRAL SEQUENCE OF A DEFORMATION

3.1. **A deformation of a complex.** Let

$$(V^\bullet, \partial_0) : 0 \rightarrow V^0 \xrightarrow{\partial_0} \dots \xrightarrow{\partial_0} V^n \rightarrow 0 \quad (3.1)$$

be a complex of real vector spaces.

Denote by  $\mathcal{O}$  the ring of germs at the origin of real analytic functions of one variable  $t$ . By a *deformation of*  $(V^\bullet, \partial_0)$  we shall understand a complex

$$(\mathcal{V}^\bullet, \partial) : 0 \rightarrow \mathcal{V}^0 \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathcal{V}^n \rightarrow 0 \quad (3.2)$$

of  $\mathcal{O}$ -modules together with a fixed isomorphism between the fiber  $\mathcal{V}^\bullet \otimes \mathcal{O}/t\mathcal{O}$  of  $(\mathcal{V}^\bullet, \partial)$  at the point  $t = 0$  and  $(V^\bullet, \partial_0)$ .

3.2. **The spectral sequence.** Given a deformation (3.2) of a complex (3.1), we consider a short exact sequence of complexes

$$0 \rightarrow \mathcal{V}^\bullet \xrightarrow{t} \mathcal{V}^\bullet \xrightarrow{q} \mathcal{B}^\bullet \rightarrow 0. \quad (3.3)$$

Here  $t : \mathcal{V}^\bullet \rightarrow \mathcal{V}^\bullet$  is the multiplication by  $t$ ,  $\mathcal{B}^\bullet$  is the quotient complex  $\mathcal{B}^\bullet = \mathcal{V}^\bullet/t\mathcal{V}^\bullet$  and  $q$  is the quotient map.

The exact sequence (3.3) induces a long exact sequence of cohomology

$$\dots \rightarrow H^m(\mathcal{V}^\bullet) \xrightarrow{t} H^m(\mathcal{V}^\bullet) \xrightarrow{q} H^m(\mathcal{B}^\bullet) \xrightarrow{r} H^{m+1}(\mathcal{V}^\bullet) \rightarrow \dots, \quad (3.4)$$

which may be rewritten as an exact couple

$$\begin{array}{ccc} H^\bullet(\mathcal{V}^\bullet) & \longrightarrow & H^\bullet(\mathcal{V}^\bullet) \\ & \swarrow & \searrow \\ & H^\bullet(\mathcal{B}^\bullet) & \end{array} \quad (3.5)$$

According to the standard rules the latter generates a (Bockstein) spectral sequence  $(\mathcal{E}_r^i, d_r)$ . To describe it explicitly set

$$\mathcal{Z}_r^i = \{s \in \mathcal{V}^i : \partial s \in t^r \mathcal{V}^{i+1}\}. \quad (3.6)$$

Then

$$\mathcal{E}_r^i = \begin{cases} V^i & \text{for } r = 0, \\ \mathcal{Z}_r^i / (t\mathcal{Z}_{r-1}^i + t^{1-r}\partial\mathcal{Z}_{r-1}^{i-1}) & \text{for } r > 0, \end{cases} \quad (3.7)$$

and the differential

$$d_r : \mathcal{E}_r^i \rightarrow \mathcal{E}_r^{i+1} \quad (3.8)$$

is the homomorphism induced by the action of  $t^{-r}\partial$  on  $\mathcal{Z}_r^i$ . Then  $H^\bullet(\mathcal{E}_r^\bullet, d_r) \simeq \mathcal{E}_{r+1}^\bullet$ , i.e. one gets a spectral sequence.

Note that the sequence  $(\mathcal{E}_r^i, d_r)$  is completely determined by (3.3).

#### 4. DEFORMATION OF THE TORSION OF A FINITE DIMENSIONAL COMPLEX

In this section we consider a one parameter family  $(V, \partial_t)$  of finite dimensional Euclidean complexes. This family can be considered as a deformation of a complex and, according to the previous section, gives rise to a spectral sequence. We define the torsion of this spectral sequence. Finally we prove a theorem by Farber (cf. [Fa, Theorem 6.6]) which describes the asymptotic for  $t \rightarrow 0$  of the torsion  $\rho(t)$  of the complex  $(V, \partial_t)$ .

In this section we essentially follow [Fa].

##### 4.1. A family of Euclidean complexes. Let

$$(V^\bullet, \partial_t) : 0 \rightarrow V^0 \xrightarrow{\partial_t} \dots \xrightarrow{\partial_t} V^n \rightarrow 0 \quad (4.1)$$

be a one parameter family of complexes of finite dimensional Euclidean vector spaces. We shall assume that the operators

$$\partial_t : V^i \rightarrow V^i$$

depend analytically on a parameter  $t$  varying within an interval  $(-\varepsilon, \varepsilon)$ . That means that  $\partial_t$  may be represented as a convergent power series

$$\partial_t = \partial_0 + t\partial_1 + \dots \quad (4.2)$$

with coefficients in  $\text{End}(V)$ .

**4.2. The germ complex.** Set  $\mathcal{V}^i = \mathcal{O} \otimes_{\mathbb{R}} V^i$  ( $0 \leq i \leq n$ ). The family (4.1) can be understood as a single complex

$$(\mathcal{V}^\bullet, \partial) : 0 \rightarrow \mathcal{V}^0 \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathcal{V}^n \rightarrow 0 \quad (4.3)$$

of  $\mathcal{O}$ -modules, where the differential  $\partial : \mathcal{V}^\bullet \rightarrow \mathcal{V}^\bullet$  is given by

$$\partial s(t) = \partial_t s(t) \quad (s(t) \in \mathcal{V}). \quad (4.4)$$

The complex (4.3) is called the *germ complex* (cf. [Fa, Section 2.5]).

**4.3. The parameterized spectral decomposition.** Let  $\partial_t^*$  ( $-\varepsilon < t < \varepsilon$ ) be the adjoint of  $\partial_t$  with respect to the Euclidean structure on  $V^\bullet$ . Set  $\Delta_t = \partial_t \partial_t^* + \partial_t^* \partial_t$ . For  $0 \leq i \leq n$ , we denote by  $\Delta_t^i$  the restriction of  $\Delta_t$  on  $V^i$ .

By a theorem of Rellich [Re, §1, Theorem 1] (see also [Ka, Ch. 7, Theorem 3.9]), there exists a family of analytic curves  $\phi_j^i(t) \in \mathcal{V}^i$  ( $1 \leq j \leq \dim V^i$ ) and a sequence of real valued analytic functions  $\lambda_j^i(t) \in \mathcal{O}$  ( $1 \leq j \leq \dim V^i$ ) such that for any value of  $t$  the numbers  $\{\lambda_j^i(t)\}$  represent all the repeated eigenvalues of  $\Delta_t^i$  and  $\{\phi_j^i(t)\}$  form a complete orthonormal basis of corresponding eigenvectors of  $\Delta_t^i$ .

As the operators  $\Delta_t^i$  are non negative for any  $t$ , the functions  $\lambda_j^i(t)$  depend only on  $t^2$ .

Suppose that  $\lambda_j^i(t)$  and  $\phi_j^i(t)$  have been numerate so that there exist integers  $0 = N_0^i \leq N_1^i \leq \dots \leq N_{m_i}^i \leq N_{m_i+1}^i = \dim V^i$  such that

- (1)  $\lambda_j^i(t) = t^{2k} \bar{\lambda}_j^i(t)$  with  $\bar{\lambda}_j^i(0) \neq 0$  for  $N_k^i + 1 \leq j \leq N_{k+1}^i$ ,  $0 \leq k \leq m_i - 1$ ;
- (2)  $\lambda_j^i(t) \equiv 0$  for  $j \geq N_{m_i}^i + 1$ .

**4.4. The torsion as a function of the parameter.** For each  $t \in (-\varepsilon, \varepsilon)$ , we shall denote by  $\rho(t)$  the torsion of the complex  $(V^\bullet, \partial_t)$ . The following lemma follows directly from (1.8).

**Lemma 4.5.** *The function  $\rho(t)$  admits an asymptotic expansion for  $t \rightarrow 0$  of the form*

$$\log \rho(t) = \frac{1}{2} \sum_{i=0}^n (-1)^i i \sum_{j=1}^{N_{m_i}^i} \log \bar{\lambda}_j^i(0) + \left( \sum_{i=1}^n (-1)^i i \sum_{k=1}^{m_i-1} k (N_{k+1}^i - N_k^i) \right) \log(t) + o(1). \quad (4.5)$$

Now our goal is to express the right hand side of (4.5) in terms of the spectral sequence of deformation (4.3).

4.6. Let  $\mathcal{V}_k^i$  ( $0 \leq k \leq m_i$ ) denote the submodule of  $\mathcal{V}^i$  generated by the set  $\{\phi_j^i \mid N_k^i + 1 \leq j \leq N_{k+1}^i\}$ . Since the operator  $\partial_t$  commutes with  $\Delta_t$  for any  $t$ , we get  $\partial \mathcal{V}_{k-1}^i \mathcal{V}_k^{i+1}$ . Then the equality

$$\langle \Delta_t s(t), s(t) \rangle = \|\partial_t s(t)\|^2 + \|\partial_t^* s(t)\|^2 \quad (4.6)$$

implies the following lemma.

**Lemma 4.7.** *If  $N_k^i + 1 \leq j \leq N_{k+1}^i$ , then*

$$\partial \phi_j^i \in t^k \mathcal{V}_k^{i+1}. \quad (4.7)$$

**4.8. The Hodge spectral sequence.** Now we shall give a Hodge theoretical description of the spectral sequence  $(\mathcal{E}_r^i, d_r)$  associated with the deformation (4.3).

For  $r \geq 0$  and  $0 \leq i \leq n$ , set

$$H_r^i = \text{span} \left\{ \phi_j^i(0) : N_r^i + 1 \leq j \leq \dim V^i \right\}_1 V^i. \quad (4.8)$$

By Lemma 4.7,  $\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(t) \in \mathcal{Z}_r^i$  for any numbers  $a_j^i \in \mathbb{R}$ . Hence, there is a natural function  $\Phi_r : H_r^\bullet \rightarrow \mathcal{E}_r^\bullet$  which maps  $\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(0)$  to the image of  $\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(t) \in \mathcal{Z}_r^i$  in  $\mathcal{E}_r^i = \mathcal{Z}_r^i / (t\mathcal{Z}_r^i + t^{1-r}\partial\mathcal{Z}_{r-1}^{i-1})$ .

**Lemma 4.9.** *For any  $r \geq 0$ , the map  $\Phi_r : H_r^\bullet \rightarrow \mathcal{E}_r^\bullet$  is injective.*

*Proof.* Suppose that

$$\Phi_r \left( \sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(0) \right) = 0.$$

Then

$$\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(t) = t\alpha(t) + t^{1-r}\partial\beta(t),$$

where  $\alpha \in \mathcal{Z}_r^i$ ,  $\beta \in \mathcal{Z}_{r-1}^{i-1}$ . Hence,

$$\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(0) = t^{1-r}\partial\beta(t) \Big|_{t=0}. \quad (4.9)$$

By Lemma 4.7,

$$t^{1-r}\partial\beta(t) \Big|_{t=0} \in \text{span} \left\{ \phi_j^i(0) : N_{r-1}^i + 1 \leq j \leq N_r^i \right\}. \quad (4.10)$$

Hence, (4.9) implies that  $\sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(0) = 0$ .  $\square$

By Lemma 4.7, there is a map  $\delta_r : H_r^\bullet \rightarrow H_r^\bullet$  defined by

$$\delta_r : \sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(0) = t^{-r}\partial \sum_{j=N_r^i+1}^{\dim V^i} a_j^i \phi_j^i(t) \Big|_{t=0}. \quad (4.11)$$

Clearly,  $\delta_r^2 = 0$ .

Let  $\delta_r^* : H_r^\bullet \rightarrow H_r^\bullet$  be the adjoint of  $\delta_r$ . Using (4.11) and Lemma 4.7, we see that

$$(\delta\delta^* + \delta^*\delta)\phi_j^i(0) = \begin{cases} 0, & \text{if } j > N_{r+1}^i; \\ \bar{\lambda}_j^i(0), & \text{if } N_r^i + 1 \leq j \leq N_{r+1}^i. \end{cases} \quad (4.12)$$

From (4.8), (4.12), we get

$$H_{r+1}^\bullet = \{h \in H_r^\bullet : (\delta\delta^* + \delta^*\delta)h = 0\} = \{h \in H_r^\bullet : \delta_r h = \delta_r^* h = 0\}. \quad (4.13)$$

The following proposition is equivalent to Theorem 3.3 of [KK]. A particular case of this result was proved by Forman ([Fo, Theorem 6]).

**Proposition 4.10.** *For all  $r \geq 0$ ,*

- (i) *The map  $\Phi_r : H_r^\bullet \rightarrow \mathcal{E}_r^\bullet$  is an isomorphism.*
- (ii)  *$\delta_r = \Phi_r^{-1} d_r \Phi_r$ .*

*Proof.* Following [KK], we shall prove the proposition by induction on  $r$ .

The case  $r = 0$  is obvious. For the inductive step, assume we have proven that  $\Phi_r : H_r^\bullet \rightarrow \mathcal{E}_r^\bullet$  is an isomorphism and  $\delta_r = \Phi_r^{-1} d_r \Phi_r$ . Then, by (4.13) and the finite dimensional Hodge theory,  $\mathcal{E}_{r+1}^\bullet$  is isomorphic to

$$\{h \in H_r^\bullet : \delta_r h = \delta_r^* h = 0\} = H_{r+1}^\bullet.$$

Then Lemma 4.9 implies that  $\Phi_{r+1} : H_{r+1}^\bullet \rightarrow \mathcal{E}_{r+1}^\bullet$  is an isomorphism. From the definition of  $d_r$  and  $\delta_r$ , we obtain  $\Phi_{r+1} \delta_{r+1} = d_{r+1} \Phi_{r+1}$ , completing the induction step and the theorem.  $\square$

**Corollary 4.11.** *For any  $0 \leq i \leq n$  and  $r \geq 0$ ,*

$$\dim \mathcal{E}_r^i = \dim V^i - N_r^i. \quad (4.14)$$

As another corollary we obtain the following result by Farber [Fa, Theorem 1.6]

**Proposition 4.12.** *The spectral sequence  $(\mathcal{E}_r^\bullet, d_r)$  stabilizes and the limit term  $\mathcal{E}_\infty^\bullet$  is isomorphic to the cohomology  $H^\bullet(V^\bullet, \partial_t)$  for a generic point  $t$ .*

**4.13. The torsion of the spectral sequence.** As a subspace of  $V^\bullet$  the vector space  $H_r^\bullet$  ( $r \geq 0$ ) inherits the Euclidean metric. We denote by  $\rho_r$  the torsion of the complex  $(H_r^\bullet, \delta_r)$  corresponding to this metric. Note that, by Proposition 4.12,  $\rho_N = 1$  for sufficiently large  $N$ .

**Definition 4.14.** The *torsion of spectral sequence*  $(\mathcal{E}_r^\bullet, d_r)$  is the product

$$\rho^{ss} = \rho_0 \rho_1 \cdots \rho_N, \quad (4.15)$$

where  $N$  is a sufficiently large number.

*Remark 4.15.* The torsion of the spectral sequence of a deformation was defined by Farber [Fa, Section 6.5] in slightly different terms. Note that the torsion of a spectral sequence, as it is defined in [Fa], corresponds, in our terms, to the product  $\rho_1 \rho_2 \cdots \rho_N$ .

4.16. **Farber theorem.** Using [Fa, Proposition 6.3], one can easily see that the following theorem is equivalent to [Fa, Theorem 6.6].

**Theorem 4.17.** *The function  $\rho(t)$  admits an asymptotic expansion for  $t \rightarrow 0$  of the form*

$$\log \rho(t) = \log \rho^{ss} + \left( \sum_{i=0}^n (-1)^i i \sum_{r \geq 1} r (\dim \mathcal{E}_r^i - \dim \mathcal{E}_{r+1}^i) \right) \log(t) + o(1). \quad (4.16)$$

*Proof.* By (1.8), (4.12), we obtain

$$\log \rho_r = \frac{1}{2} \sum_{i=0}^n (-1)^i i \sum_{j=N_r^i+1}^{N_{r+1}^i} \log \bar{\lambda}_j^i(0). \quad (4.17)$$

From (4.5), (4.14), (4.15) and (4.17), we get (4.16).  $\square$

## 5. DEFORMATION OF A FILTERED COMPLEX

In this section we apply Theorem 4.17 to a deformation of a filtered complex. The results of this section are closely related to [Fo, Theorem 9].

5.1. **The spectral sequence of a filtration.** Suppose that

$$(V^\bullet, \partial) : 0 \rightarrow V^0 \xrightarrow{\partial} \dots \xrightarrow{\partial} V^n \rightarrow 0 \quad (5.1)$$

is a filtered complex of real vector spaces with an increasing filtration

$$0 = F_0 V^\bullet \subset F_1 V^\bullet \subset \dots, \quad V^\bullet = \bigcup_{i \geq 0} F_i V^\bullet. \quad (5.2)$$

Denote by  $(E_r^{p,q}, d_r)$  ( $r \geq 0$ ) the spectral sequence of this filtration and set  $E_r^i = \bigoplus_{p+q=i} E_r^{p,q}$ . Then  $(E_r^i, d_r)$  is also a spectral sequence. We shall need the following description of  $(E_r^i, d_r)$  (cf. [BT, §14]).

Let  $i : \bigoplus_{p=0}^\infty F_p V^\bullet \rightarrow \bigoplus_{p=0}^\infty F_p V^\bullet$  be the map induced by the inclusions  $F_p V^\bullet \hookrightarrow F_{p+1} V^\bullet$  ( $p \geq 0$ ). Consider the exact sequence of complexes

$$0 \rightarrow \bigoplus_{p=0}^\infty F_p V^\bullet \xrightarrow{i} \bigoplus_{p=0}^\infty F_p V^\bullet \xrightarrow{j} B^\bullet \rightarrow 0. \quad (5.3)$$

The complex  $B^\bullet$  is isomorphic to the associated graded complex  $gr V^\bullet$  of  $V^\bullet$ .

In a standard way, the exact sequence (5.3) gives rise to a spectral sequence, which is isomorphic to  $(E_r^i, d_r)$ . It follows, that  $(E_r^i, d_r)$  is completely determined by (5.3).

**5.2. The Rees complex.** We shall construct a complex  $(\mathcal{V}^\bullet, \bar{\partial})$  of  $\mathcal{O}$ -modules as follows

$$\mathcal{V}^i = \left\{ \sum_{m=0}^N v_m t^m : v_i \in F_m V^i, N \in \mathbb{N} \right\}, \quad \mathcal{V}^\bullet = \bigoplus_{\square=\neq}^{\times} \mathcal{V}^\square,$$

$$\bar{\partial} \sum_{m=0}^N v_p t^p = \sum_{m=0}^N (\partial v_p) t^p. \quad (5.4)$$

Note that the fiber of  $\mathcal{V}^\bullet$  at the point  $t = 0$  is isomorphic to the associated graded complex  $grV^\bullet$  of  $V^\bullet$ . Hence,  $(\mathcal{V}^\bullet, \bar{\partial})$  is a deformation of  $grV^\bullet$  in the sense of Section 3.1.

As in Section 3.2, we construct a short exact sequence of complexes

$$0 \rightarrow \mathcal{V}^\bullet \xrightarrow{t} \mathcal{V}^\bullet \xrightarrow{a} \mathcal{B}^\bullet \rightarrow 0, \quad (5.5)$$

which induces a spectral sequence  $(\mathcal{E}_r^i, \bar{d}_r)$ .

Note that, as a real vector space,  $\mathcal{V}^\bullet$  is isomorphic to  $\bigoplus F_k V^\bullet$  and the multiplication by  $t$  corresponds under this isomorphism to the inclusion  $i : \bigoplus F_k V^\bullet \hookrightarrow \bigoplus F_k V^\bullet$ . Hence, the exact sequences (5.3) and (5.5) are isomorphic. Then so are the spectral sequences  $(E_r^i, d_r)$  and  $(\mathcal{E}_r^i, \bar{d}_r)$ .

**5.3. Deformation of the torsion.** Suppose that the dimensions of the spaces  $V^1, \dots, V^n$  are finite. Then there exists  $k \in \mathbb{N}$  such that  $F_j V^\bullet = V^\bullet$  for any  $j \geq k$ . Let  $g^{V^0}, \dots, g^{V^n}$  be Euclidean metrics on  $V^0, \dots, V^n$ . With these assumptions we shall present an equivalent description of deformation (5.4).

Equip  $V = \bigoplus_{i=0}^n V^i$  with the metric  $g^V = \bigoplus_{i=0}^n g^{V^i}$ , which is the orthogonal sum of the metrics  $g^{V^0}, \dots, g^{V^n}$ .

Let  $\Pi_j : V^\bullet \rightarrow F_j V^\bullet$  ( $0 \leq j \leq k$ ) be the orthogonal projection. For  $t > 0$ , set

$$A_t = \sum_{j=1}^k t^j (\Pi_j - \Pi_{j-1}). \quad (5.6)$$

The operator  $A_t$  is invertible for any  $t \neq 0$ .

Define

$$\partial_t = \begin{cases} A_t^{-1} \partial A_t, & \text{for } t \neq 0; \\ \sum_{j=1}^k (\Pi_j - \Pi_{j-1}) \partial (\Pi_j - \Pi_{j-1}), & \text{for } t = 0. \end{cases} \quad (5.7)$$

As in Section 4.1, the family of complexes  $(V_\bullet, \partial_t)$  may be considered as a single complex  $(\tilde{V}, \tilde{\partial})$  of  $\mathcal{O}$ -modules. Denote by  $A : \tilde{V}^\bullet \rightarrow \mathcal{V}^\bullet$  the

map defined by the formula

$$A \sum_{i=0}^N x_i t^i = \sum_{i=0}^N (Ax_i) t^i. \quad (5.8)$$

Then  $A$  is an isomorphism of complexes of  $\mathcal{O}$ -modules. Hence, the spectral sequence of deformation  $(\tilde{V}^\bullet, \tilde{\partial})$  is isomorphic to  $(E_r^i, d_r)$ . Let  $\rho^{ss}$  denote the torsion of this spectral sequence (cf. Definition 4.14). From Theorem 4.17, we get

**Proposition 5.4.** *Let  $\rho(t)$  be the torsion of the complex  $(V^\bullet, \partial_t)$  associated to the metric  $g^V$ . Then  $\rho(t)$  admits an asymptotic expansion for  $t \rightarrow 0$  of the form*

$$\log \rho(t) = \log \rho^{ss} + \left( \sum_{p,q \geq 0} (-1)^{p+q} (p+q) \sum_{r \geq 1} r (\dim E_r^{p,q} - \dim E_{r+1}^{p,q}) \right) \log(t) + o(1). \quad (5.9)$$

**5.5. Deformation of the metric.** Fix  $m \in \mathbb{Z}$  and let  $g_t^V$  ( $t > 0$ ) be the metric on  $V$  defined by the formula

$$g_t^V(x, y) = g^V(t^{2m} A_t^{-2} x, y). \quad (5.10)$$

We denote by  $\partial_t^*$  the adjoint of  $\partial$  with respect to the metric  $g_t^V$ . Then  $\partial_t^* = A_t^2 \partial^* A_t^{-2}$ . Set

$$\Delta_t = \partial \partial_t^* + \partial_t^* \partial. \quad (5.11)$$

Let  $\tilde{\partial}_t^*$  be the adjoint of  $\partial_t$  with respect to the metric  $g^V$ . Then  $\tilde{\partial}_t^* = A_t \partial^* A_t^{-1}$ . Set

$$\tilde{\Delta}_t = \partial_t \tilde{\partial}_t^* + \tilde{\partial}_t^* \partial_t. \quad (5.12)$$

It is easy to see that  $\tilde{\Delta}_t = A_t^{-1} \Delta_t A_t$ . Hence, by the definition of the torsion, we obtain the following lemma.

**Lemma 5.6.** *For any  $t > 0$ , the torsion of the complex  $(V^\bullet, \partial)$  associated to the metric  $g_t^V$  is equal to the torsion of the complex  $(V^\bullet, \partial_t)$  associated to the metric  $g^V$ .*

## 6. PROOF OF PROPOSITION 0.7

In this section we assume that all the critical values of  $f$  are rational.

**6.1. A filtration on the Thom-Smale complex.** Recall that the integers  $d, m$  and  $k$  were defined in Section 0.5. The Thom-Smale complex  $(C^\bullet(W^u, F), \partial)$  possesses a natural filtration

$$0 = F_0 C^\bullet(W^u, F) \supset F_1 C^\bullet(W^u, F) \supset \dots \supset F_k C^\bullet(W^u, F) = C^\bullet(W^u, F), \quad (6.1)$$

with

$$F_i C^\bullet(W^u, F) = \bigoplus_{\substack{x \in B \\ f(x) \geq \frac{m-i}{d}}} [W^u(x)]^* \otimes_{\mathbb{R}} F_x \quad (0 \leq i \leq k). \quad (6.2)$$

We denote by  $(E_r^{p,q}, d_r)$  the spectral sequence of this filtration. This spectral sequence is isomorphic to the spectral sequence of filtration (0.6).

Let  $\rho^{ss}$  denote the torsion of the spectral sequence  $(E_r^{p,q}, d_r)$ .

**6.2. Proof of Proposition 0.7.** Recall that  $\mathcal{F} \in \text{End}(C^\bullet(W^u, F))$  was defined in Section 2.9. Let  $\Pi_j : C^\bullet(W^u, F) \rightarrow F_j(C^\bullet(W^u, F))$  ( $0 \leq j \leq k$ ) be the orthogonal projection. Set  $\tau = e^{-\frac{t}{d}}$ . Then  $\tau \rightarrow 0$  as  $t \rightarrow +\infty$  and

$$e^{-2t\mathcal{F}} = \tau^{2m} \left( \sum_{j=1}^k \tau^j (\Pi_j - \Pi_{j-1}) \right)^{-2}. \quad (6.3)$$

Hence, by Proposition 5.4, Lemma 5.6 and (2.8), we see that, for  $t \rightarrow +\infty$ ,

$$\log \rho^{\mathcal{M}}(t) = \log \rho^{ss} - \left( \sum_{p,q \geq 0} (-1)^{p+q} (p+q) \sum_{r \geq 1} r (\dim E_r^{p,q} - \dim E_{r+1}^{p,q}) \right) \frac{t}{d} + o(1), \quad (6.4)$$

which is exactly Proposition 0.7.

## 7. THE RAY-SINGER METRIC AND THE RAY-SINGER TORSION

**7.1. The  $L_2$  metric on the determinant line.** Let  $(\Omega^\bullet(M, F), d^F)$  be the de Rham complex of the smooth sections of  $\bigwedge(T^*M) \otimes F$  equipped with the coboundary operator  $d^F$ . The cohomology of this complex is canonically isomorphic to  $H^\bullet(M, F)$ .

Let  $*$  be the Hodge operator associated to the metric  $g^{TM}$ . We equip  $\Omega^\bullet(M, F)$  with the inner product

$$\langle \alpha, \alpha' \rangle_{\Omega^\bullet(M, F)} = \int_M \langle \alpha \wedge * \alpha' \rangle_{g^F}. \quad (7.1)$$

By Hodge theory, we can identify  $H^\bullet(M, F)$  with the space of harmonic forms in  $\Omega^\bullet(M, F)$ . This space inherits the Euclidean product (7.1).

The  $L_2$  metric  $|\cdot|_{\det H^\bullet(M,F)}^{RS}$  on  $\det H^\bullet(M,F)$  is the metric induced by this product.

**7.2. The Ray-Singer torsion.** Let  $d^{F*}$  be the formal adjoint of  $d^F$  with respect to the metrics  $g^{TM}$  and  $g^F$ .

Set  $\Delta = d^F d^{F*} + d^{F*} d^F$  and let  $P : \Omega^\bullet(M,F) \rightarrow \text{Ker } \Delta$  be the orthogonal projection. Set  $P^\perp = 1 - P$ .

Let  $N$  be the operator defining the  $\mathbb{Z}$ -grading of  $\Omega^\bullet(M,F)$ , i.e.  $N$  acts on  $\Omega^i(M,F)$  by multiplication by  $i$ .

If an operator  $A : \Omega^\bullet(M,F) \rightarrow \Omega^\bullet(M,F)$  is trace class, we define its supertrace  $\text{Tr}_s[A]$  as in (1.6).

For  $s \in \mathbb{C}$ ,  $\text{Re } s > n/2$ , set

$$\zeta^{RS}(s) = -\text{Tr}_s [N(\Delta)^{-s} P^\perp].$$

By a result of Seeley [Se],  $\zeta^{RS}(s)$  extends to a meromorphic function of  $s \in \mathbb{C}$ , which is holomorphic at  $s = 0$ .

**Definition 7.3.** The *Ray-Singer torsion* is the number

$$\rho^{RS} = \exp\left(\frac{1}{2} \frac{d\zeta^{RS}(0)}{ds}\right). \quad (7.2)$$

**7.4. The Ray-Singer metric.** We now remind the following definition (cf. [BZ1, Definition 2.2]):

**Definition 7.5.** The *Ray-Singer metric*  $\|\cdot\|_{\det H^\bullet(M,F)}^{RS}$  on the line  $\det H^\bullet(M,F)$  is the product

$$\|\cdot\|_{\det H^\bullet(M,F)}^{RS} = |\cdot|_{\det H^\bullet(M,F)}^{RS} \cdot \rho^{RS}. \quad (7.3)$$

*Remark 7.6.* When  $M$  is odd dimensional, Ray and Singer [RS, Theorem 2.1] proved that the metric  $\|\cdot\|_{\det H^\bullet(M,F)}^{RS}$  is a topological invariant, i.e. does not depend on the metrics  $g^{TM}$  or  $g^F$ . Bismut and Zhang [BZ1, Theorem 0.1] described explicitly the dependents of  $\|\cdot\|_{\det H^\bullet(M,F)}^{RS}$  on  $g^{TM}$  and  $g^F$  in the case when  $\dim M$  is even.

**7.7. Bismut-Zhang theorem.** Let  $\nabla^{TM}$  be the Levi-Civita connection on  $TM$  corresponding to the metric  $g^{TM}$ , and let  $e(TM, \nabla^{TM})$  be the associated representative of the Euler class of  $TM$  in Chern-Weil theory.

Let  $\psi(TM, \nabla^{TM})$  be the Mathai-Quillen ([MQ, §7])  $n-1$  current on  $TM$  (see also [BGS, Section3] and [BZ1, Section IIIId]).

Let  $\nabla^F$  be the flat connection on  $F$  and let  $\theta(F, g^F)$  be the 1-form on  $M$  defined by (cf. [BZ1, Section IVd])

$$\theta(F, g^F) = \text{Tr} [(g^F)^{-1} \nabla^F g^F]. \quad (7.4)$$

Now we remind the following theorem by Bismut and Zhang [BZ1, Theorem 0.2].

**Theorem 7.8** (Bismut-Zhang). *The following identity holds*

$$\log \left( \frac{\|\cdot\|_{\det H^\bullet(M,F)}^{RS}}{\|\cdot\|_{\det H^\bullet(M,F)}^{\mathcal{M}}} \right)^2 = - \int_M \theta(F, g^F) (\nabla f)^* \psi(TM, \nabla^{TM}). \quad (7.5)$$

**7.9. Dependence on the metric.** The metrics  $\|\cdot\|_{\det H^\bullet(M,F)}^{RS}$  and  $\|\cdot\|_{\det H^\bullet(M,F)}^{\mathcal{M}}$  depend, in general, on the metric  $g^F$ . Let  $g_t^F = e^{-2tf} g^F$  and let  $\|\cdot\|_{\det H^\bullet(M,F),t}^{RS}$  and  $\|\cdot\|_{\det H^\bullet(M,F),t}^{\mathcal{M}}$  be the Ray-Singer and Milnor metrics on  $\det H^\bullet(M,F)$  associated to the metrics  $g_t^F$  and  $g^{TM}$ .

By [BZ1, Theorem 6.3]

$$\begin{aligned} \int_M \theta(F, g_t^F) (\nabla f)^* \psi(TM, \nabla^{TM}) &= \int_M \theta(F, g^F) (\nabla f)^* \psi(TM, \nabla^{TM}) \\ &\quad + 2t \operatorname{rk}(F) \int_M f e(TM, \nabla^{TM}) - 2t \operatorname{rk}(F) \operatorname{Tr}_s^B[f]. \end{aligned} \quad (7.6)$$

From (7.5) and (7.6), we get

$$\begin{aligned} \log \left( \frac{\|\cdot\|_{\det H^\bullet(M,F),t}^{RS}}{\|\cdot\|_{\det H^\bullet(M,F),t}^{\mathcal{M}}} \right)^2 &= - \int_M \theta(F, g^F) (\nabla f)^* \psi(TM, \nabla^{TM}) \\ &\quad - 2t \operatorname{rk}(F) \int_M f e(TM, \nabla^{TM}) + 2t \operatorname{rk}(F) \operatorname{Tr}_s^B[f]. \end{aligned} \quad (7.7)$$

## 8. PROOF OF PROPOSITION 0.16 AND THEOREM 0.10

8.1. For each  $t > 0$ , we equip  $\Omega^\bullet(M, F)$  with the inner product

$$\langle \alpha, \alpha' \rangle_{\Omega^\bullet(M,F),t} = \int_M \langle \alpha \wedge * \alpha' \rangle_{g_t^F}, \quad (8.1)$$

and we denote by  $|\cdot|_{\det H^\bullet(M,F),t}^{RS}$  the  $L_2$  metric on  $\det H^\bullet(M,F)$  (cf. Section 7.1) associated to this inner product.

Let  $|\cdot|_{\det H^\bullet(M,F),t}^{\mathcal{M}}$  be the Hodge-Milnor metric on  $\det H^\bullet(M,F)$  (cf. Definition 2.6) associated to the metric  $g_t^F$  on  $F$ .

From (2.5), (7.3) and (7.7), we get

$$\begin{aligned} \log \rho^{RS}(t) - \log \rho^{\mathcal{M}}(t) &= \\ &= - \frac{1}{2} \int_M \theta(F, g^F) (\nabla f)^* \psi(TM, \nabla^{TM}) - t \operatorname{rk}(F) \int_M f e(TM, \nabla^{TM}) \\ &\quad + t \operatorname{rk}(F) \operatorname{Tr}_s^B[f] + \log \left( \frac{|\cdot|_{\det H^\bullet(M,F),t}^{\mathcal{M}}}{|\cdot|_{\det H^\bullet(M,F),t}^{RS}} \right). \end{aligned} \quad (8.2)$$

Proposition 0.16 follows now from (8.2) and the following lemma:

**Lemma 8.2.** *As  $t \rightarrow +\infty$ ,*

$$\log \left( \frac{|\cdot|_{\det H^\bullet(M, F), t}^{\mathcal{M}}}{|\cdot|_{\det H^\bullet(M, F), t}^{\mathcal{RS}}} \right) = \left( \frac{n}{4} \chi(F) - \frac{1}{2} \chi'(F) \right) \log \left( \frac{t}{\pi} \right) + o(1). \quad (8.3)$$

*Proof.* Let  $d_t^{F*}$  ( $t > 0$ ) be the formal adjoint of  $d^F$  with respect to the inner product (8.1) and let  $\Delta_t = d^F d_t^{F*} + d_t^{F*} d^F$ .

Let  $\Omega_t^{\bullet, [0, 1]}(M, F)$  be the direct sum of the eigenspaces of  $\Delta_t$  associated to eigenvalues  $\lambda \in [0, 1]$ . The pair  $(\Omega_t^{\bullet, [0, 1]}(M, F), d^F)$  is a sub-complex of  $(\Omega^\bullet(M, F), d^F)$  and the inclusion induces an isomorphism of cohomology

$$H^\bullet(\Omega_t^{\bullet, [0, 1]}(M, F), d^F) \simeq H^\bullet(M, F). \quad (8.4)$$

We denote by  $\|\cdot\|_{\Omega_t^{\bullet, [0, 1]}(M, F), t}$  the norm on  $\Omega_t^{\bullet, [0, 1]}(M, F)$  determined by inner product (8.1) and by  $\|\cdot\|_{C^\bullet(W^u, F), t}$  the norm on  $C^\bullet(W^u, F)$  determined by  $g_t^F$  (cf. Section 2.3).

Recall that  $\mathcal{F} \in \text{End}(C^\bullet(W^u, F))$  was defined in Section 2.9. Clearly,

$$\|\alpha\|_{C^\bullet(W^u, F), t} = \|e^{-t\mathcal{F}}\alpha\|_{C^\bullet(W^u, F), 0}, \quad (\alpha \in C^\bullet(W^u, F)). \quad (8.5)$$

For an operator  $T : C^\bullet(W^u, F) \rightarrow \Omega_t^{\bullet, [0, 1]}(M, F)$  we denote by  $T^*$  its adjoint with respect to the norms  $\|\cdot\|_{C^\bullet(W^u, F), 0}$  and  $\|\cdot\|_{\Omega_t^{\bullet, [0, 1]}(M, F), t}$ .

In the sequel,  $\mathbf{o}(1)$  denotes an element of  $\text{End}(C^\bullet(W^u, F))$  which preserves the  $\mathbb{Z}$ -grading and is  $o(1)$  as  $t \rightarrow \infty$ .

By [BZ2, Theorem 6.9], if  $t > 0$  is large enough, there exists an isomorphism

$$e_t : C^\bullet(W^u, F) \rightarrow \Omega_t^{\bullet, [0, 1]}(M, F)$$

of  $\mathbb{Z}$ -graded Euclidean vector spaces such that

$$e_t^* e_t = 1 + \mathbf{o}(1). \quad (8.6)$$

By [BZ2, Theorem 6.11], for any  $t > 0$ , there is a quasi-isomorphism of complexes

$$P_t : \left( \Omega_t^{\bullet, [0, 1]}(M, F), d^F \right) \rightarrow \left( C^\bullet(W^u, F), \partial \right),$$

which induces the canonical isomorphism

$$H^\bullet(M, F) \simeq H^\bullet(\Omega_t^{\bullet, [0, 1]}(M, F), d^F) \simeq H^\bullet(C^\bullet(W^u, F), \partial) \quad (8.7)$$

and such that

$$P_t e_t = e^{t\mathcal{F}} \left( \frac{t}{\pi} \right)^{n/4-N/2} (1 + \mathbf{o}(1)). \quad (8.8)$$

Here  $e^{t\mathcal{F}} \left( \frac{t}{\pi} \right)^{n/4-N/2}$  denotes the operator on  $C^\bullet(W^u, F)$  which, for  $x \in B$ , acts on  $[W^u(x)]^* \otimes F_x$  by multiplication by  $e^{tf(x)} \left( \frac{t}{\pi} \right)^{n/4-\text{index}(x)/2}$ . In particular, for  $t > 0$  large enough,  $P_t$  is one to one.

Let  $P_t^*$  denote the adjoint of  $P_t$  with respect to the norms  $\|\cdot\|_{C^\bullet(W^u, F), 0}$  and  $\|\cdot\|_{\Omega_t^{\bullet, [0,1]}(M, F), t}$ . From (8.6), (8.8) we get

$$P_t P_t^* = e^{2t\mathcal{F}} \left( \frac{t}{\pi} \right)^{n/2-N} (1 + \mathbf{o}(1)). \quad (8.9)$$

Denote by  $P_t^\#$  the adjoint of  $P_t$  with respect to the norms  $\|\cdot\|_{C^\bullet(W^u, F), t}$  and  $\|\cdot\|_{\Omega_t^{\bullet, [0,1]}(M, F), t}$ . Clearly,

$$P_t^\# = P_t^* \cdot e^{-2t\mathcal{F}}. \quad (8.10)$$

Hence,

$$P_t P_t^\# = \left( \frac{t}{\pi} \right)^{n/2-N} (1 + \mathbf{o}(1)). \quad (8.11)$$

Fix  $0 \leq i \leq n$ . Let  $\sigma \in H^i(M, F)$  and let  $\omega_t \in \text{Ker } \Delta_t$  be the harmonic form representing  $\sigma$ .

Denote by  $\partial_t^\#$  the adjoint of  $\partial$  with respect to the norms  $\|\cdot\|_{C^\bullet(W^u, F), t}$  and  $\|\cdot\|_{\Omega_t^{\bullet, [0,1]}(M, F), t}$  and let  $\Pi : C^\bullet(W^u, F) \rightarrow \text{Ker}(\partial\partial_t^\# + \partial_t^\#\partial)$  be the orthogonal projection. Then  $\Pi P_t \omega_t \in C^i(W^u, F)$  corresponds to  $\sigma$  via the canonical isomorphisms (1.5), (8.7).

As  $P_t$  commutes with  $\partial$ , we see that

$$P_t \omega_t \in \text{Ker } \partial, \quad \left( \frac{t}{\pi} \right)^{n/2-i} (P_t^\#)^{-1} \omega_t \in \text{Ker } \partial_t^\#. \quad (8.12)$$

By (8.11), we get  $\left( \frac{t}{\pi} \right)^{n/2-i} (P_t^\#)^{-1} \omega_t = (1 + \mathbf{o}(1)) P_t \omega_t$ . Then (8.12) implies

$$\|\Pi P_t \omega_t\|_{C^\bullet(W^u, F), t} = \|P_t \omega_t\|_{C^\bullet(W^u, F), t} (1 + o(1)). \quad (8.13)$$

It follows from (8.11), (8.13) that

$$\|\Pi P_t \omega_t\|_{C^\bullet(W^u, F), t} = \left( \frac{t}{\pi} \right)^{n/4-i/2} \|\omega_t\|_{\Omega_t^{\bullet, [0,1]}(M, F), t} (1 + o(t)). \quad (8.14)$$

By (8.14) and by the definitions of the metrics  $|\cdot|_{\det H^\bullet(M, F), t}^{\mathcal{M}} \cdot |\cdot|_{\det H^\bullet(M, F), t}^{RS}$ , we obtain (8.3).  $\square$

The proof of Proposition 0.16 is completed.

Theorem 0.10 follows now from Proposition 0.16, Lemma 0.4 and Proposition 0.7.

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