EQUIVARIANT NOVIKOV INEQUALITIES

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Abstract. We establish an equivariant generalization of the Novikov inequalities which allow to estimate the topology of the set of critical points of a closed basic invariant form by means of twisted equivariant cohomology of the manifold. We apply these inequalities to study cohomology of the fixed points set of a symplectic torus action. We show that in this case our inequalities are *perfect*, i.e. they are in fact equalities.

In [N1,N2] S.P.Novikov associated to any real cohomology class $\xi \in H^1(M,\mathbb{R})$ of a closed manifold M a sequence of integers $\beta_0(\xi),\dots,\beta_n(\xi)$ (where $n = \dim M$) and then it was shown that for any closed 1-form θ on M, having non-degenerate critical points, the Morse numbers $m_p(\theta)$ satisfy the following inequalities

$$\sum_{i=0}^{p}(-1)^i m_{p-i}(\theta) \ \geq \ \sum_{i=0}^{p}(-1)^i \beta_{p-i}(\xi), \qquad p = 0,1,2,\dots,$$

where $\xi = [\theta] \ \in H^1(M,\mathbb{R})$ is the cohomology class of θ. Cf. also [Fa], page 48, where stronger inequalities are mentioned.

In this paper we obtain an equivariant generalization of the Novikov inequalities. We consider a compact G manifold M, where G is a compact Lie group, and an invariant closed 1-form θ on M. Assuming that the form θ is *basic* (cf. below) we define an equivariant generalization of the Novikov numbers. To do so we show that any closed basic invariant form determines a one parameter family of flat bundles over the Borel construction M_G, and the equivariant Novikov numbers are then defined as the *background value of the dimension of the cohomology* of this family. It turns out that these equivariant Novikov numbers depend only on the cohomology class of the form θ. Using these equivariant Novikov numbers we construct *the Novikov counting power series*, which is the first main ingredient appearing in our inequalities.

We assume that the form θ *is non-degenerate in the sense of Bott*; this means that the critical points of θ form a submanifold C such that the Hessian of θ is non-degenerate on the normal bundle to C. Each connected component Z of the critical manifold C determines the following objects: an integer $\operatorname{ind}(Z)$ (*the index*), a flat line bundle $o(Z)$ (*the orientation bundle*) and a subgroup $G_Z \subset G$ (*the stabilizer of the component*). Using these data we construct *the equivariant Morse counting power series* which combines information about the equivariant cohomology of all

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the components Z of C. Our main theorem (Theorem 1.7) states, roughly, that the equivariant Morse power series *is greater* (in an appropriate sense) than the Novikov counting series. This statement gives actually an infinite number of inequalities involving the dimensions of the equivariant cohomology of connected components of the critical set C and the equivariant Novikov numbers.

We use in this paper *equivariant cohomology twisted by an equivariant flat vector bundle*, which is crucial for our approach. On one hand, any closed invariant basic 1-form determines a one-parameter family of such bundles, which we use to define the equivariant generalizations of the Novikov numbers. On the other hand we observe, that using this kind of cohomology allows to strengthen the inequalities considerably. Very simple examples show (we were very surprised to discover this!) that applying the well-known equivariant Morse inequalities of Atiyah and Bott [AB1, Bo2] to the case when the group G is finite, one gets estimates, which are sometimes worse than the standard Morse inequalities (ignoring the group action). We are going to devote a separate paper describing in more detail what our approach gives in this classical situation; we will construct there a variety of new Morse type inequalities (labeled by the irreducible representations of the group G) and also some combinatorial analysis and comparison between them. These results are based on the results of the present paper.

As an application of the established in the present paper equivariant Novikov inequalities we obtain new relations between the Betti numbers of the set of fixed points of a symplectic torus action and the equivariant Novikov numbers. It seems to us that the main novelty of this our result is that, in contrast with the previously known bounds for the fixed points for symplectic torus actions using the equivariant cohomology (cf. [G]), our inequalities allow to find or to estimate the individual Betti numbers of the fixed points set, and not only the sum of these numbers. We also show that under certain additional assumptions (for example, if the fixed points are all isolated, cf. §5) it is possible to find an explicit expression for the numbers of fixed points of different indices in terms of the equivariant Novikov numbers. In the case of a holomorphic action of the circle on a Kähler manifold preserving the Kähler form (which was studied by E.Witten in [Wi]) we obtain even simpler expressions involving only the dimensions of the equivariant cohomology.

The proof of the main result of the paper (Theorem 1.7) is based in its main part on the Novikov type inequalities for differential forms with non-isolated zeros, obtained in [BF1,BF2].

The paper is organized as follows. In Section 1, we define the equivariant Novikov numbers, introduce necessary notations and then formulate our main result. We also discuss here some easy examples and special cases. The sections 2 - 4 are devoted to the proof of the main theorem. In Section 2, we recall some basic facts about equivariant vector bundles, and twisted equivariant cohomology, which we use. In section 3 we collected some remarks about basic 1-forms. In Section 4, we prove the equivariant Novikov inequalities. Finally, in Section 5 we describe our main application: the results about the Betti numbers of the fixed points set of a torus action on a symplectic manifold.

Some results of this paper were briefly announced in [BF3]. In [BF3] we developed also applications to the finite group case and to studying of manifolds with boundary.

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§1. The main result

In this section we formulate the main result of the paper (Theorem 1.7).

Let G be a compact Lie group and let M be a closed G-manifold. We do not demand that G is connected and do not exclude the case where G is a finite group. We shall denote by $\mathfrak{g}$ the Lie algebra of G.

1.1. Basic 1-forms. The equivariant generalization of the Novikov inequalities which we will describe in this paper, are applicable to closed 1-forms which are *basic*. Recall that a smooth 1-form θ on a G manifold M is called basic ([AB2]) if it is G-invariant and its restriction on any orbit of the action of G equals to zero. In other words, this means that the following two conditions have to be satisfied

$$g^*\theta = \theta, \quad \iota(X_M)\theta = 0 \quad \text{for any} \quad g \in G,\ X \in \mathfrak{g}. \tag{1}$$

Here X_M is the vector field on M defined by the infinitesimal action of $\mathfrak{g}$ on M and $\iota(X_M)$ denotes the interior multiplication by X_M.

Note that, if the group G is finite, then θ is basic if and only if it is G-invariant, i.e. if $g^*\theta = \theta$ is satisfied for any $g \in G$. Also, by Lemma 3.4, *if M is connected and if the set of fixed points of the action of G on M is not empty, then any closed G-invariant 1-form on M is basic.* Note also that *any exact invariant form $\theta = df$ is basic.*

Let $H^*_G(M, \mathbb{R})$ denote the equivariant cohomology of M. There exists a natural map

$$H^1_G(M, \mathbb{R}) \to H^1(M, \mathbb{R}) \tag{2}$$

(cf. [AB2]). We observe as Lemma 3.3 that *a real cohomology class $\xi \in H^*(M, \mathbb{R})$ lies in the image of this map if and only if it may be represented by a basic differential form.*

1.2. Equivariant generalization of the Novikov numbers. In section 2 we discuss in detail the notion of *equivariant flat vector bundle over a G manifold M.* We also construct *equivariant cohomology $H^*_G(M, \mathcal{F})$ twisted by an equivariant flat bundle $\mathcal{F}$*, cf. 2.7 - 2.9. It is shown in 2.2 that *any closed basic 1-form θ determines an equivariant flat line bundle $\mathcal{E}_\theta$* (with θ being its connection form). Using these constructions we define the equivariant Novikov numbers as follows.

Given an equivariant flat bundle $\mathcal{F}$ over M and a closed basic 1-form θ on M, consider the one-parameter family $\mathcal{F} \otimes \mathcal{E}_{t\theta}$ of equivariant flat bundles, where $t \in \mathbb{R}$, (*the Novikov deformation*) and consider the twisted equivariant cohomology

$$H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta}), \qquad \text{where} \quad t \in \mathbb{R}, \tag{3}$$

as a function of $t \in \mathbb{R}$. The following Lemma describes the behavior of the dimension of the cohomology of this one-parameter family.

1.3. Lemma. *For fixed $\mathcal{F}$ and i, there exists a finite subset $S \subset \mathbb{R}$ such that the dimension of the cohomology $H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta})$ is constant for $t \notin S$ and the dimension of the cohomology $H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta})$ jumps up for $t \in S$.*

The proof follows from Lemma 4.1 below if one takes into account Definition 2.9.

The subset S which appears in the Lemma, is called the *set of jump points*; the value of the dimension of $H^i_G(M, \mathcal{F} \otimes \mathcal{E}_{t\theta})$ for $t \notin S$ is called *the background value of the dimension of this family.*

1.4. Definition. *The i-dimensional equivariant Novikov number* $\beta_i^G(\xi,\mathcal{F})$ *is defined as the background value of the dimension of the cohomology of the family* $H^i_G(M,\mathcal{F}\otimes\mathcal{E}_{t\theta})$.

Here $\xi\in\operatorname{im}[H^1_G(M,\mathbb{R})\to H^1(M,\mathbb{R})]$ denotes the cohomology class of θ. By Lemma 2.3, the equivariant flat bundle $\mathcal{E}_\theta$ is determined (up to gauge equivalence) only by ξ. By Lemma 3.3, any class $\xi\in\operatorname{im}[H^1_G(M,\mathbb{R})\to H^1(M,\mathbb{R})]$ can be realized by a basic closed form and so the equivariant Novikov numbers $\beta_i^G(\xi,\mathcal{F})$ are defined for all classes in the image $\xi\in\operatorname{im}[H^1_G(M,\mathbb{R})\to H^1(M,\mathbb{R})]$.

The formal power series

$$\mathcal{N}^G_{\xi,\mathcal{F}}(\lambda)=\sum_{i=0}^{\infty}\lambda^i\beta_i^G(\xi,\mathcal{F}) \tag{4}$$

will be called *the equivariant Novikov series.*

1.5. We will assume that the given basic 1-form θ is *non-degenerate* in the sense of R.Bott [Bo1]. Recall, that this means that the points of M, where the form θ vanishes form a submanifold of M (called the *critical set* C of θ) and the *Hessian* of θ is non-degenerate on the normal bundle to C.

In order to make clear this definition, note that if we fix a tubular neighborhood N of C in M, then the monodromy of θ along any loop in N is obviously zero. Thus there exists a unique real valued smooth function f on N such that $df=\theta_{|_N}$ and $f_{|_C}=0$. The *Hessian* of θ is then defined as the Hessian of f.

Let $\nu(C)$ denote the normal bundle of C in M. Note that $\nu(C)$ may have different dimension over different connected components of C. Since Hessian of θ is non-degenerate, the bundle $\nu(C)$ splits into the Whitney sum of two subbundles

$$\nu(C)\;=\;\nu^+(C)\;\oplus\nu^-(C), \tag{5}$$

such that the Hessian is strictly positive on $\nu^+(C)$ and strictly negative on $\nu^-(C)$. Here again, the dimension of the bundles $\nu^+(C)$ and $\nu^-(C)$ over different connected components of the critical point set may be different.

For every connected component Z of the critical set C, the dimension of the bundle $\nu^-(C)$ over Z is called the *index* of Z (as a critical submanifold of θ) and is denoted by $\operatorname{ind}(Z)$.

Let $o(Z)$ denote the *orientation bundle* of $\nu^-(C)_{|_Z}$ considered as a flat real line bundle.

1.6. The equivariant Morse series. Let Z be a connected component of the critical set C. If the group G is connected, then Z is a G-invariant submanifold of M. In the general case we denote by

$$G_Z=\{g\in G|\; g\cdot Z\subset Z\} \tag{6}$$

the stabilizer of the component Z in G. Let $|G:G_Z|$ denote the index of G_Z as a subgroup of G. Since G_Z contains the connected component of the unity in G, this index is finite.

The compact Lie group G_Z acts on the manifold Z and the flat vector bundles $\mathcal{F}_{|_Z}$ and $o(Z)$ are G_Z-equivariant. Let

$$H^*_{G_Z}(Z,\mathcal{F}_{|_Z}\otimes o(Z)) \tag{7}$$

denote the *equivariant cohomology* of the flat G_Z-equivariant vector bundle $\mathcal{F}_{|_Z} \otimes o(Z)$. Consider the *equivariant Poincaré series* of Z

$$\mathcal{P}^{G_Z}_{Z,\mathcal{F}}(\lambda) \;=\; \sum_{i=0}^{\infty} \lambda^i \dim_{\mathbb{C}} H^i_{G_Z}(Z, \mathcal{F}_{|_Z} \otimes o(Z)) \tag{8}$$

and define using it the following *equivariant Morse counting series*

$$\mathcal{M}^G_{\theta,\mathcal{F}}(\lambda) \;=\; \sum_{Z} \lambda^{\mathrm{ind}(Z)} |G : G_Z|^{-1} \mathcal{P}^{G_Z}_{Z,\mathcal{F}}(\lambda), \tag{9}$$

where the sum is taken over all connected components Z of C.

The main result of the paper is the following:

1.7. Theorem. *Suppose that G is a compact Lie group, $\mathcal{F}$ is a flat G-equivariant vector bundle over a closed G-manifold M and θ is a closed non-degenerate (in the sense of Bott) basic 1-form on M. Then there exists a formal power series $\mathcal{Q}(\lambda)$ with non-negative integer coefficients, such that*

$$\mathcal{M}^G_{\theta,\mathcal{F}}(\lambda) - \mathcal{N}^G_{\xi,\mathcal{F}}(\lambda) = (1+\lambda)\mathcal{Q}(\lambda). \tag{10}$$

The theorem is proven in Section 4.

It is an interesting question under which conditions one may expect vanishing of the power series $\mathcal{Q}(\lambda)$ in (10). For the case $\xi = [\theta] = 0$, this problem was solved by Atiyah and Bott [AB1]. We show below in Theorem 5.3 that the generalized moment map in our symplectic applications (cf. §5) *is perfect*, i.e. $\mathcal{Q}(\lambda)$ vanishes in our inequalities (10).

We observe also, that our arguments work and prove perfectness in a more general situation: namely, under the conditions of Theorem 1.7, assuming additionally that G is connected and the set of critical points of a basic 1-form (as in Theorem 1.7) coincides with the fixed point set, cf. remark 5.4.

1.8. Examples. Consider first the case when G acts freely on M. Then the basic form θ defines a closed 1-form θ' on M/G (cf. [KN], Ch.XII, §1). It is straightforward to see, that in this case the inequalities of Theorem 1.7 (with $\mathcal{F}$ being the trivial line bundle) reduce to the usual Novikov inequalities with respect to the form θ' on the quotient manifold M/G. In particular, we see that in this situation the equivariant Novikov numbers $\beta^G_i(\xi, \mathcal{F})$ vanish for large i.

As another example, consider the case of *the circle action* $G = S^1$. Applying the Localization Theorem (cf. 2.11) we obtain that for large i the equivariant Novikov numbers $\beta^G_i(\xi, \mathcal{F})$ are two-periodic and coincide with the sum of even or odd, respectively, usual (i.e. non-equivariant) Novikov numbers of the fixed point set, cf. 2.12 and (56).

We shall consider now a special case of Theorem 1.7.

1.9. The case of isolated critical points. In this section we will assume that all critical points of θ are isolated. For simplicity we will assume that for any critical point $x \in C$ the action of the stabilizer $G_x = \{g \in G : \ g \cdot x = x\}$ of x on the tensor product $\mathcal{F}_{|_x} \otimes o(\{x\})$ is trivial. Note that this condition is automatically satisfied if the group G is connected.

It is clear that any two singular points of θ, belonging to the same orbit of G, have the same index. Let $m_i(\theta)$ denote *the number of orbits of critical points of θ having Morse index i*. We will show that the equivariant Novikov inequalities established in Theorem 1.7 give estimates of the numbers $m_i(\theta)$.

Denote by H^*_G the G equivariant cohomology ring of the point with complex coefficients, i.e. the cohomology of the classifying space BG of G with coefficients in $\mathbb{C}$. Set $\beta^G_i = \dim_{\mathbb{C}} H^i_G$ and let

$$\mathcal{P}^G(\lambda) = \sum_{i=0}^{\infty} \lambda^i \beta^G_i \tag{11}$$

be the Poincaré series of G. Note that, if G is a finite group, then $\mathcal{P}^G(\lambda) = 1$. Also, if H is a subgroup of G of finite index then $\mathcal{P}^H(\lambda) = \mathcal{P}^G(\lambda)$.

It is easy to see that the equivariant Morse counting polynomial (9) takes the form

$$\mathcal{M}^G_{\theta,\mathcal{F}}(\lambda) = d \cdot \mathcal{P}^G(\lambda) \sum_{i=0}^{\infty} \lambda^i m_i(\theta), \qquad \text{where} \quad d = \dim \mathcal{F}. \tag{12}$$

We observe, that the Morse counting polynomial depends in this case only on the dimension d of the flat equivariant bundle $\mathcal{F}$. However, simple examples (constructed using remark 1.8) show that the Novikov counting polynomial $\mathcal{N}^G_{\xi,\mathcal{F}}(\lambda)$ may really depend on choice of the flat bundle $\mathcal{F}$. Theorem 1.7 gives in this situation many inequalities involving the numbers $m_i(\theta)$ and the equivariant Novikov numbers.

In the case when all critical points of θ are isolated, it is more realistic to assume that G is a *finite group*. Then $\mathcal{P}^G(\lambda) = 1$ and we obtain $\mathcal{M}^G_{\theta,\mathcal{F}}(\lambda) = d \cdot \sum_{i=0}^{\infty} \lambda^i m_i(\theta)$. Thus Theorem 1.7 gives the inequalities

$$\sum_{i=0}^{p} (-1)^i m_{p-i}(\theta) \ \geq \ d^{-1} \cdot \sum_{i=0}^{p} (-1)^i \beta^G_{p-i}(\xi, \mathcal{F}), \qquad p = 0, 1, 2, \dots, \tag{13}$$

which look similar to the usual Novikov inequalities [N1], [N2], [N3] (but note that the meaning of our notation $m_i(\theta)$ is different).

§2. Twisted equivariant cohomology

In this section we review some basic facts about equivariant flat vector bundles, which we need in this paper. We are mainly interested in the pushforward construction which produces a flat vector bundle over the base space of a principal fiber bundle assuming that an equivariant flat vector bundle over the space of the fibration is given. We will apply the construction of pushforward in order to produce flat vector bundles over the Borel construction M_G.

2.1. Equivariant flat vector bundles. Let G be a compact Lie group and let M be a G-manifold. We assume that G acts smoothly on M from the left; we denote this action by

$$G \times M \to M, \qquad (g, x) \mapsto g \cdot x. \tag{14}$$

Let $\mathcal{F} \to M$ be a flat vector bundle over M and let

$$\nabla : \Omega^*(M,\mathcal{F}) \to \Omega^{*+1}(M,\mathcal{F}), \qquad \nabla^2 \;=\; 0 \tag{15}$$

denote the covariant derivative defined by the flat structure on $\mathcal{F}$. We will suppose that G acts smoothly on the space $\mathcal{F}$ as well

$$G \times \mathcal{F} \to \mathcal{F}, \qquad (g,\xi) \mapsto g\cdot\xi$$

and that this action is compatible with the action of G on M in the sense that the projection of the bundle $\mathcal{F} \to M$ is G-equivariant. We will also assume that the action $g : \mathcal{F}_x \to \mathcal{F}_{g\cdot x}$ is linear for any $g \in G$ and $x \in M$.

Now, the group G acts on the space of smooth sections $\Omega^0(M,\mathcal{F})$ by the formula

$$(g\cdot s)(x) = g\cdot s(g^{-1}\cdot x). \tag{16}$$

We will consider also the similar action of G on the spaces of smooth forms $\Omega^*(M,\mathcal{F})$ on M with values in $\mathcal{F}$.

For any element X of the Lie algebra $\mathfrak{g}$ of G, we will denote by $\mathcal{L}^{\mathcal{F}}(X)$ the corresponding infinitesimal action

$$\mathcal{L}^{\mathcal{F}}(X)s = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \exp(\varepsilon X)\cdot s, \qquad \text{where} \quad s \in \Omega^0(M,\mathcal{F}). \tag{17}$$

Also, for $X \in \mathfrak{g}$, we denote by X_M the vector field on M defined by the infinitesimal action of $\mathfrak{g}$ on M.

Now we arrive at the main definition:

A flat bundle $\mathcal{F} \to M$ as above, supplied with an action of G on $\mathcal{F}$ which is linear on fibers and which is compatible with the projection, is called G-equivariant flat vector bundle if the following two conditions are satisfied:

$$g\circ\nabla \;=\; \nabla\circ g \;: \Omega^0(M,\mathcal{F}) \to \Omega^1(M,\mathcal{F}) \tag{18}$$

for any $g \in G$, and

$$\nabla_{X_M} \;=\; \mathcal{L}^{\mathcal{F}}(X) :\; \Omega^0(M,\mathcal{F}) \to \Omega^0(M,\mathcal{F}) \tag{19}$$

for any $X \in \mathfrak{g}$.

Note that condition (19) determines completely the covariant derivative in the directions tangent to the orbits of G in terms of the action of G on M and on $\mathcal{F}$. It implies that *any flat section s of $\mathcal{F}$ (determined locally over an open set $U \subset M$) is invariant, i.e. $s(gx) = gs(x)$ for all $x \in U$ and for any $g \in G$ close to the unit $e \in G$.*

In particular, it follows from the above remarks that condition (19) implies that the *flat bundle $\mathcal{F}$ is trivial over any orbit of G, if we assume that G acts freely.*

Note also, that if we have two different flat connections ∇ and ∇' satisfying (19) then the difference $\nabla - \nabla'$ is a 1-form with values in $\operatorname{End}(\mathcal{F})$ which vanishes on vectors tangent to the orbits of G, i.e. it is basic.

It is clear that if the group G is connected, then condition (18) follows from (19). Conversely, if G is a finite group, the second condition (19) carries no information. Observe, that for a general compact group G (which is neither connected, nor finite) the conditions (18) and (19) are independent.

2.2. Example. The following example is crucial for our study of the Novikov inequalities. Let M be a G manifold and let θ be a closed G-invariant real 1-form on M. The last condition means that $g^*\theta = \theta$ for all $g \in G$. Consider the flat vector bundle determined by the form θ. Namely, let

$$\mathcal{E}_\theta \;=\; M \times \mathbb{C} \tag{20}$$

with the G action coming from the factor M and with the flat connection

$$\nabla \;=\; d \;+\; \theta \wedge \cdot \;\; . \tag{21}$$

Then this flat bundle always satisfy (18) and it satisfies (19) if and only if the form θ is basic.

2.3. Lemma. *The equivariant flat bundle $\mathcal{E}_\theta$ (cf. above) considered up to equivariant gauge equivalence, depends only on the cohomology class $\xi = [\theta] \in H^1(M, \mathbb{R})$.*

Proof. We may assume that M is connected; otherwise, one can consider the situation over the connected components separately.

Suppose that $\theta_1 \;=\; \theta \;+\; df$ is another basic 1-form, where $f : M \to \mathbb{R}$ is a smooth function. Then for any $X \in \mathfrak{g}$ the derivative $X_M(f)$ vanishes and so the function f must be constant on the connected components of the orbits of G. Since df is G-invariant we obtain that for any $g \in G$ there is a constant $c(g)$ such that $f(gx) = f(x) + c(g)(x)$ for all $x \in M$.

We claim that $c(g)$ must be identically zero. If there is $g \in G$ with $c(g) > 0$ then we obtain $f(gx) > f(x)$ for all $x \in M$ which is impossible. To see this, consider a point $x \in M$ such that f achieves at x it maximum on the compact set Gx. The case when $c(g) < 0$ may be treated similarly.

The arguments above prove that the function f is G invariant. Now we may define the gauge equivalence $F : \mathcal{E}_{\theta_1} \to \mathcal{E}_\theta$ as the operator of multiplication by the function $F = e^f$. It is equivariant, commutes with the projection and satisfies $\nabla \circ F \;=\; F \circ \nabla_1$, where $\nabla_1 \;=\; d \;+\; \theta_1 \wedge \cdot$. □

2.4. Remarks. We first note, that given two equivariant flat vector bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ over a G-manifold M, the Whitney sum $\mathcal{F}_1 \oplus \mathcal{F}_2$ and the tensor product $\mathcal{F}_1 \otimes \mathcal{F}_2$ (considered together with their standard flat connections and the G actions) are equivariant flat vector bundles.

Another easy observation: if $f : M_1 \to M_2$ is an equivariant map between G manifolds and if $\mathcal{F} \to M_2$ is an equivariant flat bundle over M_2, then the induced bundle $f^*\mathcal{F}$ over M_1 is also an equivariant flat bundle.

2.5. The pushforward construction. We will suppose here that *the action of G on M is free.* Then (by the Gleason lemma, cf. [Hs], page 9) the quotient space $B = M/G$ is a smooth manifold and the quotient map $q : M \to B$ is a locally trivial fibration. We will see that in this case any equivariant flat vector bundle over M determines canonically a flat vector bundle over B.

Let $\mathcal{F}$ be a G-equivariant flat vector bundle over M and let $\mathcal{S}(\mathcal{F})$ denote the locally constant sheaf of flat sections of $\mathcal{F}$. Denote by $q_*\mathcal{S}(\mathcal{F})$ the direct image of sheaf $\mathcal{S}(\mathcal{F})$. Since $\mathcal{F}$ is trivial over any connected component of any orbit of G (cf. above), and since the projection q is locally trivial, it follows that $q_*\mathcal{S}(\mathcal{F})$ is also a locally constant sheaf over B; its fiber over a point $b \in B$ can be identified with $\oplus_i \mathcal{F}_{m_i}$, where m_i's are some representatives of the connected components of $q^{-1}(b)$. Let $q_*\mathcal{F}$ denote the flat bundle corresponding to the sheaf $q_*\mathcal{S}(\mathcal{F})$.

Observe that the group G acts naturally on the direct image $q_*\mathcal{F}$ and this action is compatible with the flat structure. The action of the connected component G_0 of the unit element $e \in G$ is obviously trivial. Thus, the flat bundle $q_*\mathcal{F}$ splits into a direct sum of its flat subbundles corresponding to different irreducible representations of the finite group G/G_0. The most important for us will be the subbundle corresponding to the trivial representation; we will denote it by

$$q_*^G(\mathcal{F}). \tag{22}$$

It is the subbundle of G-invariants in $q_*\mathcal{F}$.

In particular, if G is connected, then $q_*^G\mathcal{F} = q_*\mathcal{F}$.

We will say that the flat bundle $q_*^G\mathcal{F}$ is the pushforward of the bundle $\mathcal{F}$.

Remark. We refer the reader to [BL] for a detailed study of equivariant sheaves. In particular, it is shown in [BL] that the functor $\mathcal{F} \mapsto q_*^G\mathcal{F}$ from the category of G-equivariant flat vector bundles on M to the category of flat vector bundles on $B = M/G$ is an equivalence of categories. The inverse functor is the pull-back q^* of vector bundles.

2.6. Example. We will consider now the simplest situation leading to equivariant flat vector bundles.

Let $\rho : G \to \mathrm{End}\,(V)$ be a linear representation of G which is trivial on the component G_0 of the unit element $e \in G$. Consider $\mathcal{F} = M \times V$ with the trivial connection and with the diagonal G action. It is an equivariant flat vector bundle. This bundle is trivial as a flat vector bundle but *it is not necessarily trivial as an equivariant flat vector bundle.* Thus, we may use the previous construction of pushforward to produce a flat bundle over $B = M/G$ starting from any linear representation of G/G_0.

Consider, for example, the case when $G = \mathbb{Z}_2$ acting in the standard way $x \mapsto -x$ on the sphere S^n. Let $V = \mathbb{C}$ be the unique non-trivial representation of G. Then the construction above produces the non-trivial flat line bundle over the real projective space P^n (the Möbius band).

2.7. Equivariant cohomology twisted by an equivariant flat bundle. Let M be a G manifold (the action is not supposed to be free), and let $\mathcal{F}$ be an equivariant flat vector bundle over M. Our purpose now is to define equivariant cohomology $H_G^*(M, \mathcal{F})$.

In a non-precise way we may describe our construction as follows. Let $EG \to BG$ be the universal principal bundle. Then we may consider the induced flat equivariant bundle $p^*\mathcal{F}$ over $EG \times M$, where $p : EG \times M \to M$ is the projection. Now, we want to form the pushforward $q_*^G p^*\mathcal{F}$, where $q : EG \times M \to EG \times_G M = M_G$ is the projection; it is a flat vector bundle over the Borel's quotient M_G. Now we define the equivariant cohomology $H_G^*(M, \mathcal{F})$ as the cohomology of M_G twisted by the flat vector bundle $q_*^G p^*\mathcal{F}$.

Unfortunately, we cannot really apply the construction described in the previous paragraph since our category is the category of smooth finite dimensional manifolds and the universal principal bundle $EG \to BG$ is usually infinite dimensional. Instead, we will use finite dimensional approximations to the universal principal bundle.

Consider any finite dimensional principal G-bundle $\pi : E \to B$ (with a right action of G on E) over a smooth base manifold B, and the corresponding mixing diagram

$$\begin{array}{ccccc} E & \longleftarrow & E\times M & \xrightarrow{\ p\ } & M \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle q} & & \Big\downarrow \\ B & \xleftarrow{\ \alpha\ } & E\times_G M & \longrightarrow & M/G. \end{array} \tag{23}$$

Here the action on the middle term $E\times M$ is the diagonal one $g(e,m) = (eg^{-1}, gm)$ for all $g\in G$ and $e\in E$, $m\in M$; it is a free action, since the action on E is free; q denotes the quotient map determined by this action of G.

Given a flat equivariant bundle $\mathcal{F}$ over M, we may first form the induced bundle $p^*\mathcal{F}$ over $E\times M$ (cf. 2.3) and then apply the pushforward construction (cf. 2.4) to obtain the flat bundle $q_*^G p^*\mathcal{F}$ over $E\times_G M$; the last flat bundle we will sometimes denote by $\mathcal{F}^E$ for short. Hence, the cohomology

$$H^i(E\times_G M, \mathcal{F}^E) \tag{24}$$

is defined (as the cohomology of $E\times_G M$ with coefficients in the flat bundle $\mathcal{F}^E$). We will show that for fixed i this cohomology does not depend on the choice of the principal bundle $\pi : E\to B$, provided that this principal bundle sufficiently closely approximates the universal bundle.

Let n be an integer. A principal G-bundle $\pi : E \to B$ is called *n-acyclic* if the reduced complex cohomology $\tilde{H}^j(E, \mathbb{C})$ vanishes for all $j\le n$.

2.8. Lemma. *Let M be a G manifold and let $\mathcal{F}$ be an equivariant flat vector bundle over M. Let $E\to B$ and $E'\to B'$ be two n-acyclic principal G-bundles. Denote by $\mathcal{F}^E$ and $\mathcal{F}^{E'}$ the corresponding flat vector bundles on $E\times_G M$ and $E'\times_G M$ respectively, defined as explained above. Then*

$$H^i(E\times_G M, \mathcal{F}^E) = H^i(E'\times_G M, \mathcal{F}^{E'}) \quad \text{for all} \quad i = 0,1,\dots,n-1. \tag{25}$$

Proof. Set $E'' = E\times E'$ and let G act on E'' by $g : (e,e') \mapsto (e\cdot g, e'\cdot g)$. Then E'' is an n-acyclic principal G-bundle. It is enough to show that $H^i(E''\times_G M, \mathcal{F}^{E''}) = H^i(E\times_G M, \mathcal{F}^E)$ for any $i = 0,1,\dots,n-1$.

The natural projection $r : E''\times_G M \to E\times_G M$ defines a locally trivial fibration whose fiber is isomorphic to E'. Also $\mathcal{F}^{E''} = r^*\mathcal{F}^E$ as flat vector bundles. Hence, the restriction of $\mathcal{F}^{E''}$ on the fibers of r is a trivial flat bundle. Thus, the reduced cohomology $\tilde{H}^*(E''\times_G M, \mathcal{F}^{E''})$ may be calculated by means of the spectral sequence of the fibration r. The initial term of this spectral sequence is given by

$$E_2^{pq} \;=\; H^p(E\times_G M, \tilde{\mathcal{H}}^q(E', \mathbb{C})\otimes \mathcal{F}^E). \tag{26}$$

Here $\tilde{\mathcal{H}}^q(E',\mathbb{C})$ is viewed as a local system of coefficients over $E\times_G M$ determined by the fibration r, whose fiber over a point $x\in E\times_G M$ is equal to the q-th reduced cohomology of the fiber $r^{-1}(x)\simeq E'$ and the product $\tilde{\mathcal{H}}^q(E',\mathbb{C})\otimes\mathcal{F}^E$ is understood as the tensor product of local coefficient systems.

The lemma follows now from the fact that E' is n-acyclic. □

We can give now the main definition:

2.9. Definition. *Let M be a G manifold and let $\mathcal{F}$ be an equivariant flat vector bundle over M. We define the twisted equivariant cohomology*

$$H^i_G(M,\mathcal{F}) \tag{27}$$

as the usual cohomology with coefficient in a flat vector bundle $H^i(E\times_G M,\mathcal{F}^E)$, where $E\to B$ is any $(i+1)$-acyclic principal G-bundle.

By Lemma 2.8 this definition does not depend on the choice of the $(i+1)$-acyclic principal bundle E.

2.10. Note that the twisted equivariant cohomology $H^*_G(M,\mathcal{F})$ has a natural structure of a *graded module* over the *graded equivariant cohomology ring* $H^*_G(M) = H^*_G(M,\mathbb{C})$. Also, the map α of the mixing diagram (23) induces a homomorphism of graded rings

$$\alpha^*: H^*_G \ \to\ H^*_G(M) \tag{28}$$

and so the twisted equivariant cohomology $H^*_G(M,\mathcal{F})$ is naturally a graded module over the cohomology ring $H^*_G = H^*_G(pt) = H^*(BG)$. It is well known that this ring is a polynomial ring:

$$H^*_G = \mathbb{C}[u_1,\dots,u_l] \tag{29}$$

with l generators of even degree. Here l is the rank of G, i.e. the dimension of a maximal torus $T\subset G$. If $G=T$ is a torus, these generators are all of dimension 2.

2.11. The Localization Theorem. One may formulate versions of the Localization Theorem for twisted equivariant cohomology. We will mention one such statement here since we will need it in the present paper. The proof repeats the well-known arguments (which can be found in [Hs] or [AB2]) and so it will be skipped.

We will assume in this subsection that *G is a torus.*

Let

$$M^G = \{x\in M:\ g\cdot x = x \text{ for any } g\in G\}$$

denote the set of fixed points of the action of G on M. The inclusion $i: M^G\to M$ defines a homomorphism

$$i^*: H^*_G(M,\mathcal{F}) \to H^*_G(M^G,\mathcal{F}_{|_{M^G}}) \tag{30}$$

of H^*_G-modules. The Localization Theorem states that *the kernel and cokernel of i^* are torsion modules. In particular,*

$$\operatorname{rank} H^*_G(M,\mathcal{F}) = \operatorname{rank} H^*_G(M^G,\mathcal{F}_{|_{M^G}}) = \dim_{\mathbb{C}} H^*(M^G,\mathcal{F}_{|_{M^G}}), \tag{31}$$

where the rank is understood over the field of fractions of the cohomology ring H^*_G. In the second equality we used that

$$H^*_G(M^G,\mathcal{F}_{|_{M^G}}) \ =\ H^*(M^G,\mathcal{F}_{|_{M^G}})\otimes_{\mathbb{C}} H^*_G$$

as H^*_G-modules.

2.12. Example. As an example consider the case of *the circle action $G=S^1$*. Applying the Localization Theorem we obtain that the kernel and cokernel of the map (30) are finitely generated torsion modules over $H^*_G=\mathbb{C}[u]$ and so they are finite dimensional as vector spaces. It follows that

$$\dim_{\mathbb{C}} H^i_G(M,\mathcal{F}) \ =\ \sum_{j\equiv i \mod 2} \dim H^j(M^G,\mathcal{F}_{|_{M^G}}) \tag{32}$$

for large values of the dimension i.

§3. Some remarks about basic 1-forms

In this section we will review some simple (and, probably, well-known) properties of basic forms which will be used in the paper.

Let us recall that a 1-form $\theta \in \Omega^1(M,\mathbb{R})$ on a G manifold M is called *basic* if it is G invariant and its restriction on any orbit of the action of G vanishes.

3.1. As the first remark let us mention the construction of *descent*. Suppose that the action of G on M is *free* and that θ is a basic 1-form on M. Then (by the Gleason lemma, cf. [Hs], page 9) the quotient space $B = M/G$ is a smooth manifold and the quotient map $q : M \to B$ is a locally trivial fibration. As explained in [KN, Ch. XII §1], there exists a unique 1-form $\overline{\theta}$ on B such that $q^*\overline{\theta} = \theta$. The form $\overline{\theta}$ is closed if θ is closed.

We will say that the form θ *descents* to $\overline{\theta}$.

3.2. Now we are going to mention the precise relation between the construction of descent and the pushforward construction of section 2.5.

Suppose again that G acts freely on M and let $q : M \to B$ denote the quotient map. Suppose that $\mathcal{E}$ is a line bundle over B supplied with two flat connections ∇_1 and ∇_2. The difference $\nabla_1 - \nabla_2$ is a closed 1-form on B. Consider the induced vector bundle $q^*\mathcal{E}$ over M; the flat connections ∇_1 and ∇_2 *determine uniquely the equivariant flat connections* $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ on $q^*\mathcal{E}$, respectively. Then, we claim that *the difference* $\tilde{\nabla}_1 - \tilde{\nabla}_2$ *is a closed basic 1-form on* M *which descents to the form* $\nabla_1 - \nabla_2$ *on* B.

Observe, that in the above situation the pushforward of the flat equivariant bundle $(q^*\mathcal{E}, \tilde{\nabla}_\nu)$ is the flat bundle $(\mathcal{E}, \nabla_\nu)$, where $\nu = 1, 2$.

Next we will consider the question which cohomology classes in $H^1(M,\mathbb{R})$ may be represented by a closed basic 1-form. Here we will not assume that G acts freely.

3.3. Lemma. *Let M be a G-manifold, where G is a compact Lie group. A cohomology class $\xi \in H^1(M,\mathbb{R})$ can be represented by a closed basic 1-form if and only if it belongs to the image of the homomorphism*

$$H^1_G(M,\mathbb{R}) \to H^1(M,\mathbb{R}). \tag{33}$$

Proof. Let $\pi : E \to B$ be a smooth 2-acyclic (cf. 2.7) principal G-bundle. The space $E \times_G M$ has the structure of a locally trivial fibration over the space B with fiber M. The inclusion of the fiber $i : M \to E \times_G M$ induces the homomorphism

$$i^* : H^1(E \times_G M) = H^1_G(M) \to H^1(M) \tag{34}$$

which can be identified with the homomorphism (33). We will denote by $p : E \times M \to M$ the projection.

If θ a closed basic 1-form on M then $p^*\theta$ is a closed basic 1-form on $E \times M$; by the construction of descent (cf. 3.1) it determines a closed 1-form $\overline{\theta}$ on $E \times_G M$ such that $q^*\overline{\theta} = p^*\theta$. Clearly, $i^*\overline{\theta} = \theta$. Thus, we have $[\overline{\theta}] \in H^1(E \times_G M) = H^1_G(M)$ and $i^*[\overline{\theta}] = [\theta]$, so $[\theta]$ belongs to the image of (34).

Conversely, if a cohomology class $\xi \in H^1(M,\mathbb{R})$ belongs to the image of (33) then there exists $\overline{\xi} \in H^1(E \times_G M)$ with $i^*\overline{\xi} = \xi$. We may realize $\overline{\xi}$ by a closed 1-form $\overline{\theta}$; then $\theta = i^*\overline{\theta}$ is a basic 1-form on M realizing ξ. $\square$.

We finish this section with the following useful lemma, which shows that in many applications any G invariant 1-form is automatically basic.

3.4. Lemma. *Let M be a connected G-manifold. Assume that the fixed points set of the action of G on M is not empty. Then any closed G-invariant 1-form on M is basic.*

Proof. Let θ be a closed G-invariant 1-form on M. We have to show that

$$\iota(X_M)\theta = 0 \tag{35}$$

for any $X \in \mathfrak{g}$. Clearly, it is enough to prove (35) for those $X \in \mathfrak{g}$ for which the one parameter subgroup $\{\exp(tX)|\ t \in \mathbb{R}\}$ generated by $X \in \mathfrak{g}$, is compact.

Fix $X \in \mathfrak{g}$ such that the subgroup $\{\exp(tX)|\ t \in \mathbb{R}\}$ of G is compact. For each $x \in M$ we denote by

$$\Gamma_x = \{\exp(tX) \cdot x|\ t \in \mathbb{R}\} \tag{36}$$

its orbit under the action of $\{\exp(tX)|\ t \in \mathbb{R}\}$. Since θ is G-invariant, (35) is equivalent to

$$\int_{\Gamma_x} \theta = 0 \quad \text{for any} \quad X \in \mathfrak{g}, \quad x \in M. \tag{37}$$

The fixed points set of the action $\{\exp(tX)|\ t \in \mathbb{R}\}$ on M is supposed to be not empty. Hence, each orbit Γ_x is free homotopic to a point. Since θ is closed, this implies (37). □

§4. Proof of Theorem 1.7

Throughout this section we will assume that we are in the situation of Theorem 1.7, i.e. we have a closed G manifold M, a flat equivariant vector bundle $\mathcal{F}$ over M and a closed basic 1-form θ on M which is non-degenerate in the sense of Bott. We will denote by $\xi \in H^1(M, \mathbb{R})$ the cohomology class of θ. As shown in 2.2, the form θ determines a flat equivariant line bundle $\mathcal{E}_\theta$ over M; in fact, we will consider a one-parameter family of flat equivariant bundles $\mathcal{F} \otimes \mathcal{E}_{t\theta}$ over M, where $t \in \mathbb{R}$ is a real parameter.

First we are going to obtain an approximate version of Theorem 1.7 (cf. Proposition 4.4).

Consider a principal smooth G bundle $E \to B$ over a smooth manifold B; here G acts on E from the right. As in subsections 2.7 - 2.9, we will consider the family of flat bundles

$$(\mathcal{F} \otimes \mathcal{E}_{t\theta})^E \ = \ \mathcal{F}^E \otimes \mathcal{E}^E_{t\theta}, \quad t \in \mathbb{R} \tag{38}$$

over $E \times_G M$, which are obtained form the family of flat equivariant bundles $\mathcal{F} \otimes \mathcal{E}_{t\theta}$ over M by first inducing flat bundles over $E \times M$ and then performing the pushforward construction 2.5 with respect to the projection $E \times M \to E \times_G M$.

4.1. Lemma-Definition. *Consider the dimension of the cohomology*

$$d(t) \ = \ \dim H^i(E \times_G M, \mathcal{F}^E \otimes \mathcal{E}^E_{t\theta}) \tag{39}$$

as a function of $t \in \mathbb{R}$. Then there exists a finite subset $S \subset \mathbb{R}$ such that $d(t) = d_0 =$ const for all $t \notin S$ and $d(t) > d_0$ for $t \in S$. We will call d_0 the i-dimensional E-equivariant Novikov number and will denote it by $\beta_i(\xi, \mathcal{F}; E)$; this number depends only on the cohomology class ξ of θ by Lemma 2.3.

Proof. By 3.2 there exists a closed 1-form $\tilde{\theta}$ on $E \times_G M$ such that for all t the bundle $\mathcal{E}^E_{t\theta}$ coincides (as a flat bundle) with the flat line bundle over $E \times_G M$ with the connection $d + t\tilde{\theta} \wedge \cdot$. The Lemma follows now from Lemma 1.3 of [BF1]. □

4.2. Observe that by Lemma 3.3, any class ξ in the image of the map $H^1_G(M,\mathbb{R}) \to H^1(M,\mathbb{R})$ can be realized by a closed basic form and so the E-equivariant Novikov numbers $\beta_i(\xi,\mathcal{F};E)$ are defined for all such classes.

The *E-equivariant Novikov polynomial* $\mathcal{N}^E_{\xi,\mathcal{F}}(\lambda)$ associated to a cohomology class $\xi \in \operatorname{im}[H^1_G(M,\mathbb{R}) \to H^1(M,\mathbb{R})]$ and to a G-equivariant flat vector bundle $\mathcal{F}$ is defined by the formula

$$\mathcal{N}^E_{\xi,\mathcal{F}}(\lambda) = \sum_{i=0}^{k} \lambda^i \beta_i(\xi,\mathcal{F};E), \qquad k = \dim_{\mathbb{C}}(E \times_G M). \tag{40}$$

4.3. The E-equivariant Morse polynomial. Let Z be a connected component of the critical set C of θ and let $G_Z = \{g \in G : \ g\cdot Z \subset Z\}$ denote the stabilizer of the component Z. Let $|G:G_Z|$ denote the index of G_Z as a subgroup of G. Note that G_Z contains the connected component of the unity in G (in particular, if G is connected then $G_Z = G$). Hence $|G:G_Z|$ is finite.

Recall that in Section 1.5 we associated to each connected component Z the flat line bundle $o(Z)$ over Z and an integer $\operatorname{ind}(Z)$. Let $\mathcal{F}_{|_Z}$ denote the restriction of $\mathcal{F}$ to Z. Clearly, $o(Z)$ and $\mathcal{F}_{|_Z}$ are G_Z-equivariant flat bundles.

The principal bundle E may be considered as a principal G_Z bundle over E/G_Z. Applying the induction (cf. 2.4) and then pushforward (cf. 2.5) to the flat equivariant bundles $\mathcal{F}_{|_Z}$ and $o(Z)$ we obtain flat vector bundles $\mathcal{F}^E_{|_Z}$ and $o(Z)^E$ over $Z^E = E \times_{G_Z} M$. The numbers

$$\beta^E_i(Z,\mathcal{F}_{|_Z} \otimes o(Z)) = \dim_{\mathbb{C}} H^i(Z^E, \mathcal{F}^E_{|_Z} \otimes o(Z)^E), \qquad i = 0,1,\dots,\dim Z^E \tag{41}$$

are called the *E-equivariant Betti numbers* of Z. Here $H^i(Z^E,\mathcal{F}^E_{|_Z} \otimes o(Z)^E)$ denotes the cohomology of Z^E with coefficients in the flat vector bundle $\mathcal{F}^E_{|_{Z^E}} \otimes o(Z)^E$.

Consider now the *twisted E-equivariant Poincaré polynomial* of Z

$$\mathcal{P}^E_{Z,\mathcal{F}}(\lambda) = \sum \lambda^i \beta^E_i(Z,\mathcal{F}_{|_Z} \otimes o(Z)) \tag{42}$$

and define using it the following *E-equivariant Morse counting polynomial* of θ

$$\mathcal{M}^E_{\theta,\mathcal{F}}(\lambda) = \sum_Z \lambda^{\operatorname{ind}(Z)} |G:G_Z|^{-1} \mathcal{P}^E_{Z,\mathcal{F}}(\lambda), \tag{43}$$

where the sum is taken over all connected components Z of C.

The following result can be considered as an approximate version of Theorem 1.7.

4.4. Proposition. *There exists a polynomial $\mathcal{Q}(\lambda)$ with non-negative integer coefficients, such that*

$$\mathcal{M}^E_{\theta,\mathcal{F}}(\lambda) - \mathcal{N}^E_{\xi,\mathcal{F}}(\lambda) = (1+\lambda)\mathcal{Q}(\lambda). \tag{44}$$

Proof. Let $p: E \times M \to M$ denote the projection and let $q: E \times M \to E \times_G M$ denote the quotient map (as in the mixing diagram (23)). It is shown in section 3.1 that there exists a smooth closed 1-form θ^E on $E \times_G M$ such that $p^*\theta = q^*\theta^E$

From the construction of θ^E it is clear that it is non-degenerate in the sense of Bott assuming that θ is non-degenerate.

The proof of Proposition 4.4 is based on the non-equivariant Novikov-Bott inequalities, established by the authors in [BF1, Theorem 0.3] and in [BF2, Theorem 4], applied to the form θ^E. We shall explain the details.

Let C denote the critical set of θ. Then the critical set C^E of θ^E coincides with $E \times_G C \subset E \times_G M$. If Z is a connected component of C, then

$$Z^E = E \times_{G_Z} Z \subset E \times_G M \tag{45}$$

is a connected component of C^E.

Recall that in Section 1.5 we associated to a connected component Z of C the bundle $o(Z)$ and the number $\operatorname{ind}(Z)$. Let $o(Z^E)$ and $\operatorname{ind}(Z^E)$ be the corresponding objects associated to the component Z^E of C^E. Simple calculations (cf. [AB1, Proposition 1.5]) show that $\operatorname{ind}(Z) = \operatorname{ind}(Z^E)$ and $o(Z^E) = o(Z)^E$. It follows that the *E-equivariant Betti numbers* $\beta_i^E(Z, \mathcal{F}_{|Z} \otimes o(Z))$ of Z may be calculated by the formula

$$\beta_i^E(Z, \mathcal{F}_{|Z} \otimes o(Z)) = \dim_{\mathbb{C}} H^i(Z^E, \mathcal{F}^E{}_{|Z^E} \otimes o(Z^E)), \qquad i = 0, 1, \dots, \dim Z^E. \tag{46}$$

Here $\mathcal{F}^E{}_{|Z^E}$ is the restriction of $\mathcal{F}^E$ on Z^E and $H^i(Z^E, \mathcal{F}^E{}_{|Z^E} \otimes o(Z^E))$ denote the cohomology of Z^E with coefficients in the flat vector bundle $\mathcal{F}^E{}_{|Z^E} \otimes o(Z^E) = q_*^G p^*(\mathcal{F}_{|Z} \otimes o(Z))$.

The correspondence $Z \mapsto Z^E = Z \times_{G_Z} E$ is a surjection from the set of connected components of C to the set of connected components of C^E. Moreover the preimage of Z^E contains exactly $|G : G_Z|$ connected components of C. Hence,

$$\mathcal{M}^E_{\theta,\mathcal{F}}(\lambda) = \sum_Z \lambda^{\operatorname{ind}(Z)} \cdot \sum_{i=0}^{\dim Z^E} \lambda^i \dim_{\mathbb{C}} H^i(Z^E, \mathcal{F}^E{}_{|Z^E} \otimes o(Z^E)). \tag{47}$$

This means that the E-equivariant Morse counting polynomial of θ coincides with the Morse counting polynomial of θ^E as it is defined in [BF1,BF2]:

$$\mathcal{M}^E_{\theta,\mathcal{F}}(\lambda) = \mathcal{M}_{\theta^E,\mathcal{F}^E}(\lambda).$$

The E-equivariant Novikov polynomial $\mathcal{N}^E_{\xi,\mathcal{F}}(\lambda)$ by definition is equal to the Novikov polynomial associated to the cohomology class of θ^E in $H^1(M^E, \mathbb{R})$ (cf. [BF1,BF2]). The proposition follows now from Theorem 0.3 of [BF1] (see also [BF2, Theorem 4]). □

4.5. Proof of Theorem 1.7. Let

$$\mathcal{M}^G_{\theta,\mathcal{F}}(\lambda) \;=\; \sum \alpha_i \lambda^i, \qquad \mathcal{N}^G_{\xi,\mathcal{F}}(\lambda) \;=\; \sum \beta_i \lambda^i \tag{48}$$

and denote $\gamma_i = \alpha_i - \beta_i$. Then Theorem 1.7 states that

$$\gamma_p - \gamma_{p-1} + \gamma_{p-2} - \dots + (-1)^p \gamma_0 \geq 0 \tag{49}$$

for all $p = 0, 1, 2, \dots$. For any p this inequality involves only equivariant cohomology (of M and of the critical manifolds) of dimension less than $p+1$. Hence, each of these inequalities follows from Proposition 4.4 and from the definition of twisted equivariant cohomology (cf. Definition 2.9). This completes the proof. □

§5. Application: fixed points of a symplectic torus action

In this section we apply Theorem 1.7 to study the topology of the fixed point set of a symplectic torus action. We will see that inequalities given by Theorem 1.7 are exact in this case. This gives new relations between the Betti numbers of the fixed points set of a torus action on a symplectic manifold, and the equivariant Novikov numbers.

5.1. Assume that the n-dimensional torus $T = S^1 \times \cdots \times S^1$ acts by symplectomorphisms on a compact connected symplectic manifold (M, ω). Denote by

$$M^T \;=\; \big\{ x \in M |\ t \cdot x = x \ \text{ for any } \ t \in T \big\} \tag{50}$$

the fixed point set of the action.

Let $\mathfrak{t}$ be the Lie algebra of T. Choose $X \in \mathfrak{t}$ such that the one-parameter subgroup $\{\exp(sX)|\ s \in \mathbb{R}\}$ generated by X is dense in T. Let X_M be the vector field on M defined by the infinitesimal action of $\mathfrak{t}$ on M. Let $\iota(X_M)$ denote the interior multiplication by X_M. Set

$$\theta = \iota(X_M)\omega. \tag{51}$$

Then θ is a closed T-invariant 1-form on M. We will call θ *the generalized moment map* of the torus action.

McDuff [McD] constructed examples showing that this form may be not cohomologous to zero. Clearly, the critical set of θ coincides with the set M^T of fixed points. Hence, one can use Theorem 1.7, to estimate the Betti numbers of M^T. It is shown in [Au], cf. §§2.1, 2.2, that θ is non-degenerate in the sense of Bott, each connected component Z of M^T has even index, and the flat line bundle $o(Z)$ is trivial.

In order to satisfy the assumption of Theorem 1.7 that θ is basic, we will assume that either the fixed point set M^T is not empty (then θ is basic by Lemma 3.4), or that T is the circle (and then θ is clearly basic).

5.2. Let $\mathcal{F}$ be a T-equivariant flat vector bundle on M (for instance, one may always take the trivial line bundle for $\mathcal{F}$.) For any connected component Z of M^T we denote by

$$\mathcal{P}_{Z,\mathcal{F}}(\lambda) = \sum_{i=0}^{\dim Z} \lambda^i \dim_{\mathbb{C}} H^i(Z, \mathcal{F}_{|Z}) \tag{52}$$

the Poincaré polynomial of Z and by $\operatorname{ind}(Z)$ the index of Z considered as a critical submanifold of the form θ. $\mathcal{P}^T(\lambda)$ will denote the T-equivariant Poincaré series of a point (as in 1.9). It is well known that $\mathcal{P}^T(\lambda) = (1-\lambda^2)^{-n}$. The equivariant Morse polynomial of θ is given by the formula

$$\mathcal{M}^T_{\theta,\mathcal{F}}(\lambda) = (1-\lambda^2)^{-n} \cdot \sum_Z \lambda^{\operatorname{ind}(Z)} \mathcal{P}_{Z,\mathcal{F}}(\lambda) \tag{53}$$

where the sum is taken over all connected components of the fixed points set M^T.

Let $\xi = [\theta] \in H^1(M, \mathbb{R})$ denote the cohomology class of θ and let $\mathcal{N}_{\theta,\mathcal{F}}(\lambda)$ denote the equivariant Novikov series of $\mathcal{F}$ associated to this cohomology class, cf. 1.4.

5.3. Theorem. *In the situation described in 5.1 and 5.2, assume that either the fixed point set M^T is not empty, or T is the circle. Then the following identity holds*

$$(1-\lambda^2)^{-n}\cdot\sum_Z \lambda^{\mathrm{ind}(Z)}\mathcal{P}_{Z,\mathcal{F}}(\lambda) = \mathcal{N}^T_{\xi,\mathcal{F}}(\lambda), \tag{54}$$

where the sum on the left is taken over all connected components Z of the fixed point set M^T.

This theorem gives new relations between the homology of the fixed point set M^T and the equivariant Novikov numbers. Compare [G], [Hs].

Proof. The following arguments generalize the result of V. Ginzburg [Gi]; cf. also [AB2], page 26. Note that papers [Gi] and [AB2] deal with Hamiltonian actions, i.e. assuming that the cohomology class of θ is zero.

Choose a generic t. More precisely, using Lemma 1.3, we may find a countable subset $S\subset\mathbb{R}$ such that for $t\notin S$ the dimension of $H^i_G(M,\mathcal{F}\otimes\mathcal{E}_{t\theta})$ equals the Novikov number $\beta^T_i(\xi,\mathcal{F})$ for all i. We will assume in the sequel that $t\notin S$.

Recall from section 2.10, that the twisted equivariant cohomology

$$H^*_T(M,\mathcal{F}\otimes\mathcal{E}_{t\theta}) = \bigoplus_{i=0}^{\infty} H^i_T(M,\mathcal{F}\otimes\mathcal{E}_{t\theta})$$

has a natural structure of a graded module over the ring $H^*_T=\mathbb{C}[u_1,\dots,u_n]$. Similarly, each term of the spectral sequence E^{pq}_r (where $r=2,3,\dots$) of the fibration

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & ET\times_T M \\ & & \downarrow{\scriptstyle\pi} \\ & & BT, \end{array}$$

with coefficients in the equivariant flat vector bundle $\mathcal{F}\otimes\mathcal{E}_{t\theta}$, is a graded module over H^*_T (cf. notation (29)). Here the spectral sequence $E^{*,*}_r$ is considered with respect to the total grading: $E_r=\oplus E^n_r$, where

$$E^n_r = \bigoplus_{p+q=n} E^{pq}_r.$$

Our aim is to show that all the differentials of this spectral sequence vanish. Note that from the general properties of the spectral sequence of a fibration we know that the differentials are H^*_T-module homomorphisms.

We will deal with the *ranks* of the terms of this spectral sequence, which are defined as the dimensions over the field of rational functions

$$\tilde{H}^*_T = \mathbb{C}(u_1,\dots,u_n)$$

of their localizations $\tilde{H}^*_T\otimes_{H^*_T}E_r$; we will denote these ranks by $\mathrm{rank}\,E_r$. Note that the initial term

$$E_2 = \bigoplus H^p_T\otimes H^q(M,\mathcal{F}\otimes\mathcal{E}_{t\theta})$$

of this spectral sequence *is free* as H_T^*-module and its rank is

$$\operatorname{rank} E_2 = \sum_{i=0}^{\dim M} \dim H^i(M, \mathcal{F} \otimes \mathcal{E}_{t\theta}).$$

On the other hand, the rank of the limit term E_∞ equals to

$$\operatorname{rank} E_\infty = \sum_{i=0}^{\dim M^T} \dim H^i(M^T, \mathcal{F}|_{M^T}), \tag{55}$$

as follows from the Localization theorem, cf. 2.11. Since $\operatorname{rank} E_2 \geq \operatorname{rank} E_\infty$, we obtain the inequality

$$\sum_{i=0}^{\dim M} \dim H^i(M, \mathcal{F} \otimes \mathcal{E}_{t\theta}) \geq \sum_{i=0}^{\dim M^T} \dim H^i(M^T, \mathcal{F}|_{M^T}). \tag{56}$$

On the other hand, the non-equivariant Novikov - Bott inequalities of [BF1], Theorem 0.3 (cf. also [BF2], Theorem 4) give (by substituting $\lambda = 1$) the opposite inequality to (56).

Thus we obtain that the rank of the terms of the spectral sequence is constant $\operatorname{rank} E_r = const$. Since the initial term is free, the first nontrivial differential would reduce the rank; therefore all the differentials vanish.

As another conclusion of the above arguments we obtain the identity

$$\sum_Z \lambda^{\operatorname{ind} Z} \Big(\sum_{i=0}^{\dim Z} \lambda^i \dim H^i(Z, \mathcal{F}|_Z) \Big) = \sum_{i=0}^{\dim M} \lambda^i \dim H^i(M, \mathcal{F} \otimes \mathcal{E}_{t\theta}), \tag{57}$$

which follows from Theorem 0.3 of [BF1]. In fact, we know that the polynomial $\mathcal{Q}(\lambda)$ in Theorem 0.3 of [BF1] has nonnegative coefficients, and also we know that $\mathcal{Q}(1) = 0$; therefore we obtain that $\mathcal{Q}(\lambda)$ is identically zero.

Returning again to the spectral sequence, we obtain (by comparing the Poincaré power series of the initial term with the Poincaré series of the limit) that

$$(1 - \lambda^2)^{-n} \cdot \sum_{i=0}^{\dim M} \lambda^i \dim H^i(M, \mathcal{F} \otimes \mathcal{E}_{t\theta}) = \sum_{i=0}^{\infty} \lambda^i \dim H_T^i(M, \mathcal{F} \otimes \mathcal{E}_{t\theta}). \tag{58}$$

This, combined with (57), gives the required identity (54). $\square$

5.4. Remark. Note that a similar statement concerning *perfectness or exactness* of our equivariant Novikov inequalities (10) can be proven in a more general situation of Theorem 1.7 assuming only that the set of critical points of a basic 1-form coincides with the fixed point set of an action of a compact connected Lie group. All the above arguments can then be applied.

Next we present some easy corollaries of Theorem 5.3.

5.5. Corollary. *Let M be a symplectic manifold with a symplectic circle action. M has no fixed points if and only if all the equivariant Novikov numbers $\beta_i^T(\xi, \mathcal{F})$, where $i = 0, 1, 2 \dots$, vanish. Here $\mathcal{F}$ denotes the trivial flat line bundle over M, and $\xi = [\theta]$ is the cohomology class of the generalized moment map θ (cf. (51)).*

Proof. First we should mention that the form θ (defined by (51)) is clearly basic and so the equivariant Novikov numbers are well defined. Now one simply applies Theorem 5.3. □

Here is another simple corollary:

5.6. Theorem. *Let M be a symplectic manifold with a symplectic action of the oriented circle $T = S^1$, and let $\mathcal{F}$ be a flat equivariant bundle of rank d over M. Suppose that the fixed point set M^T is non-empty and all the odd-dimensional cohomology $H^{odd}(M^T, \mathcal{F}_{|_{M^T}})$ vanishes (note that this is automatically true if M^T consists of isolated points!) Fix a positive vector $X \in \mathfrak{t}$ and consider the 1-form $\theta = i(X_M)\omega$ as in (51). Then:*

(1) *all odd-dimensional equivariant Novikov numbers $\beta^T_{2i-1}(\xi, \mathcal{F})$ vanish;*
(2) *the even dimensional equivariant Novikov numbers increase*

$$\beta_0^T(\xi, \mathcal{F}) \leq \beta_2^T(\xi, \mathcal{F}) \leq \beta_4^T(\xi, \mathcal{F}) \leq \dots, \tag{59}$$

and they stabilize

$$\beta^T_{2i}(\xi, \mathcal{F}) = \beta^T_n(\xi, \mathcal{F}) = d \cdot \chi(M) \tag{60}$$

for $i \geq n/2$, where $n = \dim M$;
(3) *assume additionally that the set of fixed points M^T is finite. Let m_i denote the number of fixed points of T, which have index i as critical points of θ. Then*

$$m_{2i} = d^{-1}[\beta^T_{2i}(\xi, \mathcal{F}) - \beta^T_{2i-2}(\xi, \mathcal{F})] \quad \text{and} \quad m_{2i-1} = 0 \tag{61}$$

for all i. Moreover, the following symmetry relation

$$m_i = m_{n-i} \tag{62}$$

holds for all i. The total number of fixed points of T equals to

$$d^{-1}\beta^T_n(\xi, \mathcal{F}) = \chi(M). \tag{63}$$

Proof. Applying Theorem 5.3 we obtain

$$(1 - \lambda^2)^{-1} \sum_Z \lambda^{\mathrm{ind}(Z)} \mathcal{P}_{Z,\mathcal{F}}(\lambda) = \mathcal{N}^T_{\xi,\mathcal{F}}(\lambda). \tag{64}$$

Assuming that $H^{odd}(M^T, \mathcal{F}|_{M^T}) = 0$, we see that the power series on the left side of (64) has only even powers of λ. This implies that the Novikov power series $\mathcal{N}^T_{\xi,\mathcal{F}}(\lambda)$ has also only even powers of λ.

Formula (59) and the fact that the Novikov numbers stabilize (i.e. $\beta^T_{2i}(\xi, \mathcal{F}) = \beta^T_n(\xi, \mathcal{F})$ for $i \geq n/2$) follow from (64) since the left hand side of (64) is clearly a rational function of the form $(1-\lambda^2)^{-1}p(\lambda)$, where the polynomial $p(\lambda)$ is of degree $\leq n$ and has only even powers with nonnegative integral coefficients. Expanding it into a power series, we see that the *stable Novikov number* $\beta_n^T(\xi,\mathcal{F})$ equals $p(1)$, which is the same as $d\cdot\chi(M)$ (by corollary 0.4 of [BF1] and since all the indices $\mathrm{ind}(Z)$ are even).

If the fixed point set M^T consists of isolated points, then the above polynomial $p(\lambda)$ is just $\sum m_i\lambda^i$ and the formula (61) follows from (64) by comparing the coefficients. Summing up formulas (61) gives (63). Formula (62) follows by reversing the orientation of the circle and using the obvious relation $\beta_i^T(\xi,\mathcal{F})=\beta_i^T(-\xi,\mathcal{F})$ between the equivariant Novikov numbers. □

Similar statement holds for the torus actions as well.

Note also that in the Kähler case, $\xi=0$ by [Fr] and, thus, the equivariant Novikov numbers in Theorem 5.6 can be replaced by the dimensions of the equivariant cohomology in the corresponding dimension. This observation shows that in this special case our Theorem 5.6 gives the following well-known statement (cf. [AB2], page 23):

5.7. Corollary. *Suppose that an oriented circle T acts holomorphically on a Kähler manifold M of dimension n such that the Kähler form is preserved and such that the fixed point set is non-empty and consists of finitely many isolated points. Then for odd i the equivariant cohomology $H_T^i(M)$ vanishes; for even numbers i the dimensions of the equivariant cohomology $H_T^i(M)$ increase*

$$\dim H_T^0(M)\leq\dim H_T^2(M)\leq\dim H_T^4(M)\leq\cdots\leq\dim H_T^n(M) \tag{65}$$

and stabilize

$$\dim H_T^i(M)=\dim H_T^n(M) \tag{66}$$

for even $i\geq n=\dim M$. Additionally, for i even, $0\leq i\leq n$, the difference

$$\dim H_T^i(M)-\dim H_T^{i-2}(M)$$

equals to

$$m_i=\dim H^i(M), \tag{67}$$

where m_i denotes the number of fixed points of the action of the circle T having index i. The total number of fixed points of T is given by the dimension of the equivariant cohomology

$$\dim_{\mathbb{C}} H_T^n(M)=\chi(M). \tag{68}$$

Proof. As we noticed above, in the situation of Corollary 5.7 $\xi=0$ by [Fr], and so there is a Hamiltonian $f:M\to\mathbb{R}$ such that the form θ (given by (51)) is $\theta=df$. Then f is a Morse function with critical points of even indices only. The Morse theory implies that f is perfect and this gives formula (67). The other statements of Corollary follow directly from Theorem 5.5. □

Observe, that applying the Morse theory to the Hamiltonian f (as above) we find that the manifold M in Corollary 5.7 is necessarily simply connected. That is why we cannot expect to gain additional information by considering general equivariant flat vector bundles $\mathcal{F}$.

Corollary 5.7 is valid in a more general situation of Hamiltonian circle actions with isolated fixed points – we have only used the condition $\xi = 0$ in the proof (cf. also [AB2], page 23). Recall that by result of D. McDuff [McD] any symplectic circle action on a 4-dimensional manifold is Hamiltonian if it has fixed points. Also, the Corollary remains true for Lefschetz manifolds (which include the Kähler manifolds), i.e. for symplectic manifolds M such that the multiplication by ω^{n-1} defines an isomorphism $H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$. It was shown by K.Ono [O] that a symplectic action of a circle on a Lefschetz manifold is Hamiltonian iff it has fixed points, cf. also [McD].

E.Witten in [Wi] constructed Morse type inequalities which estimate the topology of the fixed point set of a circle action on a Kähler manifold. He also assumes that the action is holomorphic and preserving the Kähler form. Cf. also [MW]. A generalization of these inequalities of Witten for compact Lie groups was announced recently in [Wu]. S.Wu assumes however that the fixed points are isolated.

It would be interesting to compare the results of this paper with the information given by [Wi], [Wu].

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