Northeastern University
Department of Mathematics

Qualifying Exam in Topology

January 2010

Do the following six problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as **clear** *and* **concise** *as possible. Show all your work.*

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1. Consider the following two topologies on $\mathbb{R}^2$:
   (i) The Zariski topology, $\mathcal{T}_1$, where a base of open sets is formed by the complements of zero sets of polynomials in two variables.
   (ii) The topology $\mathcal{T}_2$, the weakest topology where all straight lines are closed sets.
   Show that $(\mathbb{R}^2, \mathcal{T}_1)$ and $(\mathbb{R}^2, \mathcal{T}_2)$ are not homeomorphic.

2. Let X be a locally connected space, and let $f\colon X \to Y$ be a quotient map. Show that Y is also locally connected.

3. (a) Define the two notions: "homotopy between two maps" and "homotopy equivalence between two topological spaces."
   (b) Give an example of topological spaces X and Y that have the same homotopy type but are not homeomorphic.
   (c) Give an example of topological spaces X and Y that have isomorphic fundamental groups but are not homotopy equivalent.
   (d) Give an example of topological spaces X and Y that have isomorphic homology groups (in all degrees) but are not homotopy equivalent.

4. Let X be the space obtained by attaching two disks, D_1 and D_2, to the circle S^1, where the first disk is attached via the map $f_1\colon \partial D_1 = S^1 \to S^1$, $f_1(z) = z^2$, and the second disk is attached via the map $f_2\colon \partial D_2 = S^1 \to S^1$, $f_2(z) = z^6$.
   (a) Use the Seifert-van Kampen theorem to compute the fundamental group $\pi_1(X, x_0)$.
   (b) Compute the homology groups $H_i(X, \mathbb{Z})$, for all $i \geq 0$.

5. Let $Y = \{(z, w) \in \mathbb{C}^2 \mid z \neq w\}$. Let $X = Y/\mathbb{Z}_2$, where the cyclic group $\mathbb{Z}_2$ acts on Y by interchanging the coordinates. Let $p\colon Y \to X$ be the projection map.
   (a) Find the fundamental group of Y.
   (b) Find the fundamental group of X.
   (c) Determine the induced homomorphim $p_\sharp\colon \pi_1(Y, y_0) \to \pi_1(X, p(y_0))$.

6. Let K be the Klein bottle—a square with opposite vertical edges identified in the same direction, and opposite horizontal edges identified in the opposite direction.
   (a) Describe 3 non-equivalent, connected covering spaces of K.
   (b) For each covering, indicate the corresponding subgroup of $\pi_1(K)$, and whether the cover is regular or not.