Northeastern University, Mathematics Department.

Probability: Qualifying Exam **January 2010**

Be sure to include your reasoning – no credit for unexplained answers.

1). Suppose that $(X, \mathcal{A}, P)$ is a probability triple, where X is the sample space, $\mathcal{A}$ is the σ-algebra of events, and P is the probability law.

i). State the property of countable additivity for this triple.

ii). Suppose that $\{A_n\}$ is an increasing sequence of events, so that $A_1 \subset A_2 \subset \cdots$ Prove that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

2). Let U be a uniform random variable on $[0, 1]$. Suppose that n trials are to be performed, and that conditional on $\{U = u\}$ these trials will be independent with a common success probability u. Compute the mean and variance of the number of successes that occur in these trials.

3). Suppose that M coins are distributed among two boxes. At each time unit one of the coins is randomly selected, removed from its box and placed in the other box. Set up this system as a Markov chain with $M + 1$ states, where the state is the number of coins in the first box at each step. Show that the chain is time reversible, and find the stationary distribution for the number of coins in the first box.

4). State the Central Limit Theorem; be sure to include any necessary hypotheses.

Suppose that $\{X_i\}$ are IID uniform random variables on the interval $[-1, 1]$. Let Z be a standard normal random variable. Find the number a so that

$$\lim_{n\to\infty} P\Big(\sum_{i=1}^{n} X_i \geq \sqrt{n}\Big) = P(Z \geq a)$$

5). Suppose that $N_1(t)$ and $N_2(t)$ are independent Poisson processes, with rates 3 and 4 respectively. Starting at an arbitrary time, compute the probability that at least two arrivals from N_1 occur before three arrivals from N_2.