Northeastern University
Mathematics Department

Qualifying Exam, Algebra 2
January 2010

(1) Let A be a commutative ring. An element $p \neq 0$ is said to be *indivisible* if it is not invertible and if the relation $p = xy$ $(x, y \in A)$ implies that either x or y belongs to (p). Assume that A has the ACC. Prove that every element $a \neq 0$ of A can be written (in at least one way) as a product of indivisible elements and of one invertible element.

(2) Determine the number of elements in the ring $\mathbb{Z}[\sqrt[3]{3}]/(3 + \sqrt[3]{3} + \sqrt[3]{9})$.

(3) Let A be a ring, M an A-module, and $f : M \to M$ an endomorphism. Prove that:
   (a) If M is artinian (i.e., M has the DCC), then there is a number n such that $M = \mathrm{Ker} f^p + \mathrm{Im}\, f^p$ for any $p \geq n$. If f is a monomorphism, then it is an isomorphism.
   (b) If M is noetherian (i.e., M has the ACC), then there is a number n such that $0 = \mathrm{Ker} f^p \cap \mathrm{Im}\, f^p$ for any $p \geq n$. If f is an epimorphism, then it is an isomorphism.
   (c) If M is of finite length, then there is a number n such that $M = \mathrm{Ker} f^p \oplus \mathrm{Im}\, f^p$ for any $p \geq n$. Also, f is an isomorphism whenever it is a monomorphism or an epimorphism.

(4) Determine all irreducible polynomials of degree 5 in $\mathbb{Z}/2[t]$.

(5) A subgroup M of an abelian group N is said to be maximal if N is the only subgroup of N properly containing M.
   (a) Prove that M is maximal if and only if its index in N is a prime number.
   (b) Prove that the groups $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$ have no maximal subgroups.