Northeastern University, Mathematics Department.

Probability: Qualifying Exam **April 2010**

Be sure to include your reasoning – no credit for unexplained answers.

1). Suppose that $(X, \mathcal{A}, P)$ is a probability triple, where X is the sample space, $\mathcal{A}$ is the σ-algebra of events, and P is the probability law.

i). State the property of countable additivity for this triple.

ii). Suppose that $\{B_n\}$ is a sequence of events with $P(B_n) = 1$ for all n. Prove that

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = 1$$

2). i). Let A_n be a sequence of events. Define $\limsup_{n\to\infty} A_n$.

ii). The Borel-Cantelli Lemma states that if $\sum_n P(A_n) < \infty$ then $P(\limsup_{n\to\infty} A_n) = 0$. Suppose that $\{X_n\}$ are Bernoulli random variables with

$$P(X_n = 1) = p_n = 1 - P(X_n = 0)$$

Suppose furthermore that $\sum_n p_n < \infty$. Compute

$$P(\lim_{n\to\infty} X_n = 0)$$

3). Suppose that r balls are randomly distributed in n boxes, so that the sample space consists of n^r equally likely elements. Let N_n denote the number of empty boxes. Compute the mean and variance of N_n.

4). State the Weak Law of Large Numbers; be sure to include any necessary hypotheses and define your notion of convergence.

5). The lifetime of a machine (in days) is an integer-valued random variable T. Assume that $T \geq 1$, and that $P(T = n) = c\,p^n$ $(n = 1, \ldots)$ where $0 < p < 1$ and $c = (1-p)/p$ is a normalization constant. Given that the machine is working after k days, what is the expected value of its lifetime?