Northeastern University Department of Mathematics

Qualifying Exam: Partial Differential Equations

January 2010

1. Consider the first-order pde:

$$u_x - 2\,u_y = \tan u \quad \text{for } x, y > 0, \qquad u = \frac{\pi}{2} \quad \text{along } x = y.$$

(a) Is there a unique solution $u(x, y)$?
(b) Find an explicit formula for $u(x, y)$.
(c) Does the solution exist for all $x, y > 0$?

2. Consider equation

$$u_{xx} + x u_{xy} + u_y = u.$$

(a) Determine where it is hyperbolic and find the characteristics.
(b) For which Cauchy data is the Cauchy Problem well-posed?

$$\begin{aligned} &i)\ u(x,0) = f(x),\ u_y(x,0) = g(x),\\ &ii)\ u(0,y) = f(y),\ u_x(0,y) = g(y),\\ &iii)\ u(1,y) = f(y),\ u_x(1,y) = g(y). \end{aligned}$$

3. Use energy methods to show uniqueness of solutions $u \in C^2(\mathbf{R}^n \times (0,\infty))$ for the damped wave equation

$$\begin{aligned} &u_{tt} - c^2\Delta u + u_t = f(x,t) \qquad \text{for } x \in \mathbf{R}, t > 0,\\ &u(x,0) = g(x), \quad u_t(x,0) = h(x) \quad \text{for } x \in \mathbf{R}. \end{aligned}$$

4. Let D be the open disk of radius 2 centered at the origin in the x,y plane and suppose $u \in C^2(D)$ satisfies

$$\begin{aligned} \Delta u &= 0 \quad \text{in } D,\\ u(2,\theta) &= 1 + \cos\theta \quad \text{for } 0 \le \theta < 2\pi. \end{aligned}$$

(a) Find the maximum value of u in the closed disk $\overline{D}$.
(b) Find the value of u at $r = 0$.

5. Consider the heat conduction problem

$$\begin{aligned} u_t - 3u_{xx} &= 2t \qquad \text{for } 0 < x < \pi,\ t > 0,\\ u(x,0) &= \sin x \quad \text{for } 0 < x < \pi,\\ u(0,t) &= t^2 = u(\pi,t) \quad \text{for } t > 0. \end{aligned}$$

(a) Find a solution in the form $u(x,t) = v(t) + w(x,t)$.
(b) Is the solution in (a) unique?