# QUALIFYING EXAM, GENERAL ALGEBRA, April 2010

1. Our field is the field of complex numbers.

   a) Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

   b) Find the Jordan basis for the matrix

$$\begin{pmatrix} 1 & 4 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. For a linear form $\phi$ on a vector space $V$ we define the linear map

$$\Delta(\phi) : \bigwedge^{i} V \to \bigwedge^{i-1} V$$

   by the formula

$$\Delta(\phi)(v_1 \wedge \ldots \wedge v_i) = \sum_{u=1}^{i} (-1)^{u+1} \phi(v_u) v_1 \wedge \ldots \wedge \hat{v}_u \wedge \ldots \wedge v_i$$

   where $\hat{v}_u$ means that the factor $v_u$ is omitted. Prove that the dual map

$$\Delta(\phi)^* : \bigwedge^{i-1} V^* \to \bigwedge^{i} V^*$$

   is the exterior multiplication by $\phi$.

3. Let $V, W$ be two vector spaces over a field $K$. Assume $dim\ V = m, dim\ W = n$. Consider the pairs $(A, v)$ where $A : V \to W$ is a linear map and $v \in V$.

   a) Assume that $v \notin Ker\ A$. Prove that there exists a number $r$, $1 \leq r \leq min(m, n)$ and bases $\{v_1, \ldots, v_m\}$, $\{w_1, \ldots, w_n\}$ of $V, W$ respectively such that $A(v_i) = w_i$ for $1 \leq i \leq r$ and $A(v_i) = 0$ for $i > r$, and $v_1 = v$.

   b) Assume that $v \in Ker\ A$. Prove that there exists a number $r$, $0 \leq r \leq min(m-1, n)$ and bases $\{v_1, \ldots, v_m\}$, $\{w_1, \ldots, w_n\}$ of $V, W$ respectively such that $A(v_i) = w_i$ for $1 \leq i \leq r$ and $A(v_i) = 0$ for $i > r$, and $v_{r+1} = v$.