

Northeastern University
(Department of Mathematics)

Analysis 1 Qualifying Exam (April 2010)

1. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a monotonically decreasing non-negative real valued function that is Riemann integrable on any finite interval $[0, T]$, $T \geq 0$. Show that the improper integral $\int_0^\infty f(x) dx$ converges if and only if the series $\sum_{k=1}^\infty f(k)$ converges.

2. Evaluate the improper integral $\int_0^1 \frac{dt}{\sqrt{t(1-t)}}$.

3. Consider the differential form $\alpha := p dq - H(p, q) dt$ in $\mathbb{R}^3 = \{(p, q, t)\}$ where $H(p, q) = \frac{1}{2}(p^2 + q^2)$.

(a) Compute the differential forms $d\alpha$, $dt \wedge d\alpha$, and $dp \wedge d\alpha$;

(b) Evaluate the integral $\int_\gamma \alpha$ along the curve $\gamma(s) := (\cos s, \sin s, s)$ where $0 \leq s \leq \pi/4$;

(c) Evaluate the integral $\int_Z d\alpha$ where Z denotes the cylinder

$$Z := \{(\cos \phi, \sin \phi, s) \mid 0 \leq \phi \leq 2\pi, 0 \leq s \leq 1\}.$$

4. Let $X = (x_{ij})_{1 \leq i, j \leq n}$ be a $n \times n$ -matrix where $n \geq 3$. Consider the function $f : (\mathbb{R}^n)^n \rightarrow \mathbb{R}$ given by the formula $f(X) := \det X$ where \det denotes the determinant of X .¹

(a) Assume that $\text{rk } X = n - 1$. Is it true that X is a critical point of f ?

(b) Characterize the set of critical points of f in terms of the rank of X .

5. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in a vector space X are called *equivalent* if there exist positive real constants $C > c > 0$ such that $\forall x \in X$, $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$. Show that any two norms in \mathbb{R}^n are equivalent.

¹Here we identify the space of $n \times n$ matrices with $(\mathbb{R}^n)^n$.