Department of Mathematics
Combinatorics Qualifying Exam
January 2010

1. Suppose that G is a graph that contains cycles of sizes 4, 6, 8 and no others. In addition, suppose that every vertex of G has degree 5.

   Prove: G has a 3-regular spanning subgraph. Be sure to state any theorems you use, including all hypotheses.

2. Let Δ denote the maximum degree of the graph G, and n the number of vertices of G.

   Prove: If

$$n > 1 + \Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)^2 + \ldots \Delta(\Delta - 1)^{k-1}$$

   then the diameter of G is greater than k.

3. A regular octagon is centered at the origin and has a vertex on the x-axis. The two reflections α in the x-axis and β in the y-axis generate a group G. Now color the eight vertices of the octagon with k colors, $r_1, \ldots, r_k$ (say), but do not distinguish two such colorings which are equivalent under the group G.

   (a) Find the cycle indicator polynomial of G.

   (b) Find the total number of different models (equivalence classes of colorings) obtained in this way.

   (c) For $k = 2$, explain how you would find the number models which have three vertices colored with r_1 and five vertices colored with r_2. You do not have to compute the actual number.

4. Let k be a positive integer, and let k be fixed. Find the general solution of the linear recurrence

$$a_n = -\binom{k}{1} a_{n-1} - \binom{k}{2} a_{n-2} - \ldots\ldots - \binom{k}{k} a_{n-k}.$$

5. Let Q be a 4-dimensional cube in $\mathbb{R}^4$, and let P be the convex hull of the midpoints of the edges of Q. What type of facets do occur in P, and how many are there of each kind?

6. Let B be a convex d-polytope contained in a hyperplane H of $\mathbb{R}^{d+1}$, let x be a point of $\mathbb{R}^{d+1}$ not contained in H, and let P be the convex $(d+1)$-polytope in $\mathbb{R}^{d+1}$ defined as the convex hull of B and the translate $x+B$ of B. Such a polytope P is called a *prism* over B, and B is called its *base*.

   Express the f-vector $(f_{-1}(P), f_0(P), \ldots, f_{d+1}(P))$ of P in terms of the f-vector of B. Here $f_i(P)$ denotes the number of i-dimensional faces of P, for $i = 0, 1, \ldots, d+1$.