Department of Mathematics
Combinatorics Qualifying Exam
January 2006

1. Consider the lattices $\Lambda_t := t\mathbb{Z}^2$ with $t > 0$ in $\mathbb{E}^2$. Let C be the square centered at the origin and with vertices at $(\pm 1, 0)$, $(0, \pm 1)$.

   (a) For which values of t is Λ_t admissible for C?

   (b) For which values of t does Λ_t pack C (that is, for which t is $\Lambda_t + C$ a packing of $\mathbb{E}^2$)? What is the density of these packings of squares?

2. Let Λ be a lattice in $\mathbb{E}^2$, and let b_1, b_2 be a basis of Λ. Then

   $$\Lambda^* := \{x \in \mathbb{E}^2 \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in \Lambda\}$$

   is known to be a lattice called the *dual lattice* of Λ. Define the vectors b_1^*, b_2^* in $\mathbb{E}^2$ by

   $$b_i \cdot b_j^* = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

   Show that b_1^*, b_2^* is a basis of Λ^*.

   (Hint: Look at $x := \lambda b_1^* + \mu b_2^* \in \Lambda$ and find λ, μ.)

3. Let p be a real polynomial of degree n. Suppose that $p(x)$ is an integer for $n+1$ consecutive integers x. Show that the difference table of p can only have integral entries. Then show that $p(x)$ must be an integer for all integers x.

4. Consider a rectangular box P in 3-space whose length, width and height are distinct. The symmetry group G of P has exactly 8 elements (including three reflections and three half-turns). Color the eight vertices of P with k colors, but do not distinguish two colorings which are equivalent under G. Answer the following questions using Polya's Theorem.

   (a) List the 8 elements of G as permutations of the vertices.

   (b) What is the total number of models (equivalence classes of colorings) obtained?

5. Let G be a simple graph on $n \geq 3$ vertices. Suppose that for all vertices $x \in G$, $G - x$ is a tree.
(a) Find $e(G)$, the number of edges of G, proving your answer.
(b) Determine G, proving your answer.

6. For a graph G, let $\alpha(G)$ denote the maximum size of an independent set of vertices in G. Suppose that G is a bipartite graph on $n(G) = n$ vertices. Prove: $\alpha(G) = n/2 \iff G$ has a perfect matching.