Department of Mathematics
Combinatorics Qualifying Exam
January 2005

1. Let $n \geq 4$. Let G be a simple graph and $\delta(G) = n(G) - 2$ (where $n(G) =$ the number of vertices of G).

   Prove: $\kappa(G) = \delta(G)$.

2. (a) State Hall's Theorem for complete matchings in bipartite graphs.

   (b) Given a family of sets $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$, a *system of distinct representatives* (SDR) is a set of distinct elements $x_1, x_2, \ldots, x_m$ such that for every i, $x_i \in A_i$.
   Prove: $\mathcal{A}$ has an SDR $\Longleftrightarrow$ for all $I \subset \{1, 2, \ldots, m\}$,

   $$|\bigcup_{i \in I} A_i| \geq |I|.$$

   Hint: Model this by a graph.

3. Let Λ be the lattice in $\mathbb{E}^3$ consisting of all integral vectors (x_1, x_2, x_3) with $x_1 + x_2 + x_3$ even.
   (a) Find the determinant of Λ, that is, $\det(\Lambda)$.
   (b) Let C be the unit cube centered at the origin and with its edges parallel to the coordinate axes. Show that Λ packs C (that is, $\Lambda + C$ is a packing of $\mathbb{E}^3$). What is the density of this packing of cubes?

4. Let Λ be a lattice in $\mathbb{E}^d$. Define the *dual* of Λ by

   $$\Lambda^* := \{x \in \mathbb{E}^d \mid x \cdot y \in \mathbb{Z} \text{ for all } y \in \Lambda\}.$$

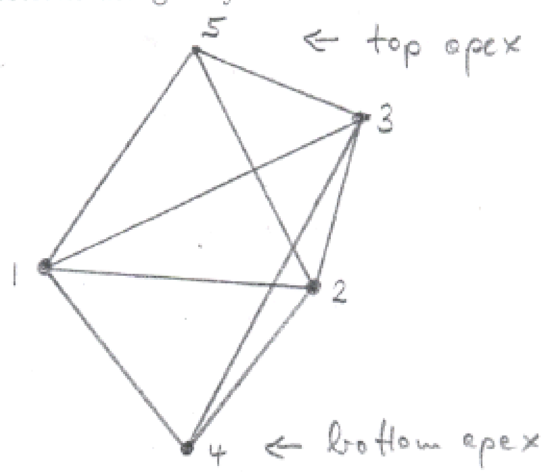
   (a) Prove that Λ^* is a lattice.
   (b) Describe how you would find Λ^* when Λ is the lattice in Problem 3. (You do not actually have to compute it.)

5. Consider the system of linear recurrences,

$$\left\{ \begin{array}{l} a_n = 2b_n - 3b_{n-1} \\ b_n = 3a_{n-1} \end{array} \right\} \quad (n \geq 1)$$

with $a_0 = 1$, $a_1 = -2$, $b_0 = 1$.
(a) Find the ordinary generating functions of a_n and b_n in closed form (as rational functions).
(b) Find explicit formulas for a_n and b_n.

6. Consider a double pyramid P obtained by gluing two regular tetrahedra along a common triangle (see the figure). The symmetry group G of P has exactly 12 elements. Color the five vertices of P with k colors, but do not distinguish two colorings which are equivalent under G. Answer the following questions using Polya's Theorem.



(a) List the 12 elements of G as permutations of the vertices $1, \ldots, 5$ of P.

(b) What is the total number of models (equivalence classes of colorings) obtained?