

Northeastern University
(Department of Mathematics)

Analysis 1 Qualifying Exam (Spring 2010)

1. Let $A = (a_{ij})_{1 \leq i, j \leq 3}$ be a 3×3 -matrix. Consider the function $f : \mathbb{R}^9 \rightarrow \mathbb{R}$ given by the formula $f(A) := \det A$ where \det is the determinant of A .¹

- (a) Find a formula for the partial derivative $\frac{\partial f}{\partial a_{ij}}$;
- (b) Compute the differential df of f at the point $A = E$ where E denotes the identity matrix.

2. Let $f : (a, b) \rightarrow \mathbb{R}$ be an $(n + 1)$ -times continuously differentiable function on (a, b) , $n \geq 2$. Assume that $f'(x_0) = \dots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, at some $x_0 \in (a, b)$.

- (a) Show that f has a local extremum at x_0 ;
- (b) Is it true that f has a local maximum at x_0 if $f^{(n)}(x_0) < 0$? Explain.

3. Consider the 1-form $\alpha := \frac{1}{2}(x dy - y dx)$ in $\mathbb{R}^2 = \{(x, y)\}$.

- (a) Compute the differential $d\alpha$. Is it true that α is a differential of a function in \mathbb{R}^3 ?
- (b) Compute the integral $\int_E \alpha$ where E is the ellipse $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$ with parameters $0 < b \leq a$. What is the area of $\{\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$?

4. Let $f : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable on $[a, b]$. Let $\mathcal{N} \subseteq [a, b]$ be a finite set of points and let $g : [a, b] \rightarrow \mathbb{R}$ be a bounded function on $[a, b]$ so that $\forall x \in [a, b] \setminus \mathcal{N}$,

$$g(x) = f(x).$$

Show that g is Riemann integrable on $[a, b]$ and $\int_a^b f(x) dx = \int_a^b g(x) dx$.

5. Let $(a_n)_{n \geq 1}$ be a sequence of complex numbers. Consider the mean $s_n := (\sum_{k=1}^n a_k)/n$. Show that if $\lim_{n \rightarrow \infty} a_n = A$ then $\lim_{n \rightarrow \infty} s_n = A$. Is it true that the convergence of $(s_n)_{n \geq 1}$ implies convergence of $(a_n)_{n \geq 1}$?

¹Here we identify the space of 3×3 matrices with \mathbb{R}^9 .