Department of Mathematics
Algebra I — Qualifying Exam
January 2010

1. Let d be a non-zero integer such that $|d|$ is a product of distinct primes. Show that
$$\mathbb{Q}(\sqrt{d}) := \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$$
is a subfield of the field $\mathbb{C}$ of complex numbers.
(If $d < 0$, then $\sqrt{d} := \sqrt{|d|}\, i$.)

2. Let p be a prime, and let $\mathbb{F}_p$ denote the field with p elements. Let $G := GL_3(p)$ be the group of all non-singular 3×3 matrices with entries in $\mathbb{F}_p$, and let $H := SL_3(p)$ be the subgroup of G consisting of all non-singular 3×3 matrices with entries in $\mathbb{F}_p$ and with determinant 1. Determine the number of elements of G and of H. Explain clearly.

3. Suppose V and W are finite-dimensional vector spaces over a field K. Use the universality property of tensor products to show that there exists a canonical isomorphism $f : V^* \otimes W \to Hom_K(V, W)$.
(Consider $\phi : V^* \times W \to Hom_K(V, W)$ given by $\phi(l, w)(v) := l(v)w$.)

4. Show that every complex $n \times n$ matrix A is similar to its transpose A^t. It helps to first consider the case when A is a Jordan block.

5. Find the Jordan canonical form for the real matrix
$$A = \begin{pmatrix} 2 & -1 & 3 \\ -4 & -1 & -4 \\ 1 & 1 & 0 \end{pmatrix}.$$

6. An $n \times n$ matrix A is called a *Hadamard matrix* if all of its entries are ± 1 and $AA^T = nI$. Show that the Kronecker product $A \otimes B$ of two $n \times n$ Hadamard matrices A and B is again a Hadamard matrix.

   (Recall that the Kronecker product $A \otimes B$ is the matrix of the tensor product of the two endomorphisms of $\mathbb{R}^n$ described by the matrices A and B, in suitable bases.)

7. Does there exist a parallelepiped P in $\mathbb{R}^4$ such that P has volume 1 and is spanned by the three vectors $v_1 = (1, 0, -1, 2)^t$, $v_2 = (-2, 1, -1, 1)^t$, and $v_3 = (1, 0, 1, 1)^t$, and a fourth vector v_4? If the parallelepiped P exists, determine the vector v_4.

8. Let V be an n-dimensional real vector space with basis $v_1, \dots, v_n$, and let Λ be the set of all integral linear combinations of $v_1, \dots, v_n$ in V. Define the subset Λ^* of the dual space V^* by

$$\Lambda^* := \{l \in V^* \mid l(v) \in \mathbb{Z} \text{ for all } v \in \Lambda\}.$$

Show that there exists a basis of V^* such that Λ^* is the set of integral linear combinations of the vectors in this basis. Describe the kernels of the vectors in this basis.

9. A real 8×8 matrix A has $2 - i$ and $3 + 4i$ among its eigenvalues, and their algebraic multiplicity is 2 (that is, they are double roots of the characteristic polynomial of A). Write down the possible generalized (real) Jordan matrices for A.