SPECIALIZATION QUALIFYING EXAM IN ALGEBRA. September 2009.

Notation: $\mathbf{C}$ =complex numbers, $\mathbf{Z}$ =integers.

**1.** Let $B = \mathbf{Z}[X]$ where $\mathbf{Z}$ is the ring of integers. Give an example of a $B$-module $M$ which is finitely generated but not a direct sum of cyclic modules.

**2.** We work over a polynomial ring $A = \mathbf{C}[X]$. Each of the matrices given below is a matrix of a homomorphism of free $A$-modules in coordinate bases. Determine the isomorphism type of the cokernel in each case (since $A$ is a PID we know this cokernel has to be isomorphic to a direct sum of cyclic modules).

a) Let $f : A^2 \to A^3$,

$$f = \begin{pmatrix} 1 & x \\ x^2 & x^2 \\ x^3 & x^3 \end{pmatrix}.$$

b) Let $g : A^2 \to A^3$,

$$g = \begin{pmatrix} x & x \\ x^2 & x \\ x^3 & x \end{pmatrix}.$$

**3.** Show an example of a commutative ring $R$, an $R$-module $N$ and an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

such that the sequence of Abelian groups we get by applying the functor $- \otimes_R N$ to our exact sequence is not exact.

**4.** Give examples of modules (over rings of your choice) with following properties. Justify your answers.

a) $M$ is projective and torsion,

b) $P$ is projective but torsion free,

c) $Q$ is injective and torsion,

d) $N$ is injective and torsion free.

**5.** Let $A$ be a PID and let $M$ be a finitely generated $A$-module. Prove that the functor $Hom_A(M, -)$ is exact if and only if $M$ is free.

**6.** Determine whether the following statements are true or false. Justify your answers by either giving a proof of a counterexample:

a) A factormodule of a free module is projective,

b) A factormodule of a torsion free module is torsion free,

c) A factormodule of an injective module is injective,

d) A factormodule of a cyclic module is cyclic.

**7.** Enumerate isomorphism classes of $\mathbf{C}[X]$-modules annihilated by $X^3(X+1)^4$.

**8.** Let us consider the following diagram of $R$-modules

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_4 & \to & M_3 & \to & M_2 & \to & M_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N_4 & \to & N_3 & \to & N_2 & \to & N_1 & \to & 0\,. \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P_4 & \to & P_3 & \to & P_2 & \to & P_1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Assume that the rows are complexes and the columns are exact. We know that only nonzero homology module of the first row is $H_2(M_\bullet)$ and only nonzero homology module is $H_3(P_\bullet)$. What can you say about homology modules $H_i(N_\bullet)$ ?