QUALIFYING EXAM IN ALGEBRA. September 2009.

1. We work over the field of real numbers. Reduce the quadratic form

$$q(x_1, \dots, x_4) = x_1^2 + x_1x_2 + x_1x_3 + x_2x_4 + x_4^2$$

to the diagonal form and calculate its signature.

2. Use universality property to prove that for a finite dimensional vector spaces V and W there is a canonical isomorphism

$$\phi : \bigwedge^2 (V \oplus W) \to \bigwedge^2 V \oplus (V \otimes W) \oplus \bigwedge^2 W.$$

3. Let V be a vector space of dimension 3 with the basis $\{e_1, e_2, e_3\}$ and let W be a vector space of dimension 3 with the basis $\{f_1, f_2, f_3\}$. Which of the following tensors are decomposable (i.e. of the form $v \otimes w$ with $v \in V$, $w \in W$).

a) $s = e_1 \otimes f_1 + e_2 \otimes f_2 - e_1 \otimes f_3$,

b) $t = e_1 \otimes f_2 + e_1 \otimes f_3 + e_2 \otimes f_2 + e_2 \otimes f_3$,

c) $e_2 \otimes f_1 + e_2 \otimes f_3$.

What is the condition for a tensor

$$v = \sum_{i=1}^{3} \sum_{j=1}^{2} a_{i,j} e_i \otimes f_j$$

from $V \otimes W$ to be decomposable.

4. Let $f : K^3 \to K^3$ be a map in Jordan canonical form having a matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Find Jordan canonical form of $f \otimes f$.

5. Let $\mathbf{R}^4$ be the Euclidean space with the scalar product givern by the usual dot product. Let V be a subspace spanned by $(1, 0, 1, 1)$, $(0,1,1,1)$, $(1, 1, 0, 1)$. Apply the Gram-Schmidt process to find the orthonormal basis of V. Find the orthogonal complement of V.

6. Describe the Abelian group $Hom_{Ab}(\mathbf{Z}/16\mathbf{Z}, \mathbf{Z}/4\mathbf{Z} \oplus \mathbf{Z}/8\mathbf{Z})$.

7. Describe all conjugacy classes of 5×5 matrices A which satisfy the equation $(A - 1)^2 A(A + 1)^2 = 0$.

8. Give examples

a) of a torsion free but not a cyclic group,

b) a group that is neither torsion free nor torsion,

c) a torsion group which is not annihilated by any integer.,