

QUALIFYING EXAM IN ALGEBRA. January 2009.

1. We work over the field of real numbers. Reduce the quadratic form

$$q(x_1, \dots, x_4) = \sum_{i=1}^4 x_i^2 + \sum_{1 \leq i < j \leq 4} x_i x_j$$

to the diagonal form and calculate its signature.

2. Use universality property to prove that for a finite dimensional vector spaces  $V$  and  $W$  there is a canonical isomorphism

$$\phi : V^* \otimes W^* \rightarrow (V \otimes W)^*.$$

3. Let  $V$  be a vector space of dimension 3 with the basis  $\{e_1, e_2, e_3\}$  and let  $W$  be a vector space of dimension 2 with the basis  $\{f_1, f_2\}$ . Which of the following tensors are decomposable (i.e. of the form  $v \otimes w$  with  $v \in V, w \in W$ ).

- a)  $s = e_1 \otimes f_1 - e_2 \otimes f_2$ ,  
b)  $t = e_1 \otimes f_2 - e_1 \otimes f_3 + e_2 \otimes f_2 - e_2 \otimes f_3$ ,  
c)  $u = e_1 \otimes f_2 - e_1 \otimes f_3 + e_2 \otimes f_2$ .

What is the condition for a tensor

$$v = \sum_{i=1}^3 \sum_{j=1}^2 a_{i,j} e_i \otimes f_j$$

from  $V \otimes W$  to be decomposable.

4. Let  $f : K^3 \rightarrow K^3$  be a map in Jordan canonical form having a matrix

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find Jordan canonical form of  $f \otimes f$ .

5. Let  $\mathbf{R}^4$  be the Euclidean space with the scalar product given by the usual dot product. Let  $V$  be a subspace spanned by  $(1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1)$ . Apply the Gram-Schmidt process to find the orthonormal basis of  $V$ . Find the orthogonal complement of  $V$ .
6. Describe explicitly a subgroup  $H$  of order 8 of the permutation group  $S_5$ .
7. Describe the Abelian group  $\text{Hom}_{\text{Ab}}(\mathbf{Z}/10\mathbf{Z}, \mathbf{Z} \oplus \mathbf{Z}/15\mathbf{Z})$ .

8. Describe all conjugacy classes of  $6 \times 6$  matrices  $A$  which satisfy the equation  $A^2(A + 1)^4 = 0$ .
9. Give the examples of groups of the following type:
- a) An infinite non-Abelian group,
  - b) An infinite torsion Abelian group,
  - c) A homomorphism of groups which is neither injective nor surjective,
  - d) A subgroup which is not normal,
10. Consider a group  $G := GL_2(\mathbf{F}_2)$  of  $2 \times 2$  non-singular matrices over a field  $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$  of two elements.
- a) Prove that  $G$  has 6 elements,
  - b) Show that  $G$  is isomorphic to the permutation group  $S_3$ .